11/12/25 T
https://t.me/webinarsKZ/3708/3717
Bob Greenyer: Dear Colleagues,
The discovery that accelerating the rotation of a rotating energy quanta/mass leads to the observed effects was proposed by Gabriel Kron in 1934 for which he won the George Montefiore Foundation prize of the University of Liege and 10,000 Belgium Francs in 1935.

Robert Forward extended the work of the earlier Thirring and Lense papers (built on relativity) from the early 1900s (see Mashhoon translations and comments) - see his 1962 paper attached. Robert L. Forward specifically addresses the physical manifestation of accelerated rotation.
He also addressed the Lense-Thirring effect with respect to energy solitons, which is applicable to both the MHD in nuclear detonations and thos in PLASMa torOIDs (Commonly referred to as PLASMOIDS) - EVOs, Charge Clusters, Energy Solitons etc.

What none of these address, is the self-organising, fractal Magneto-Hydrodynamic structures I predicted and then have established as fact from empirical data that occur in plasma and cavitation systems. These are self-feeding, self reinforcing, self organising, virial structures with a non-EM radiating boundary, inside of which the permittivity of the vacuum is changed, resulting in
1. Change to speed of light
2. Change to the relative to outside rate of flow of time
3. Change in the behaviour of charge interactions
4. Lowering of the mass/energy requirement to achieve 'black holes' in the structures that are integral to their stability.
I look forward to today's discussion.
Regards
Bob
Боб Гриньер: Уважаемые коллеги,
Открытие того, что ускорение вращения квантов энергии/массы приводит к наблюдаемым эффектам, было предложено Габриэлем Кроном в 1934 году, за что он получил премию Фонда Джорджа Монтефиоре Льежского университета и 10 000 бельгийских франков в 1935 году.

Роберт Форвард продолжил работу более ранних статей Тирринга и Лензе (основанных на теории относительности) начала 1900-х годов (см. переводы и комментарии Машхуна) — см. его статью 1962 года, прилагаемую к этому сообщению. Роберт Л. Форвард конкретно рассматривает физическое проявление ускоренного вращения.

Он также затронул эффект Лензе-Тирринга в отношении энергетических солитонов, который применим как к МГД в ядерных детонациях, так и к МГД в плазмоидах (обычно называемых плазмоидами) — ЭВО, кластерам заряда, энергетическим солитонам и т. д.

Однако ни один из этих подходов не рассматривает самоорганизующиеся фрактальные магнитогидродинамические структуры, которые я предсказал и затем подтвердил эмпирическими данными, наблюдаемые в плазменных и кавитационных системах. Это самоподдерживающиеся, самоусиливающиеся, самоорганизующиеся вириальные структуры с неэлектромагнитной излучающей границей, внутри которой изменяется диэлектрическая проницаемость вакуума, что приводит к:
1. изменению скорости света;
2. изменению относительной скорости течения времени по сравнению с внешней средой;
3. изменению поведения взаимодействий заряда;
4. снижению требований к массе/энергии для образования «черных дыр» в структурах, которые являются неотъемлемой частью их стабильности.
С нетерпением жду сегодняшней дискуссии. С уважением,
Боб

Андрей Чистолинов:

Добрый день уважаемые коллеги.
Прочитал анонс предстоящего семинара Климова-Зателепина и решил, что не могу не отреагировать на это знаменательное событие. Кстати вопрос к Климову: Анатолий Иванович, а вас устраивает, что Зателепин

постоянно прикрывается вами, свою фамилию из названия семинара не хотите забрать? Или вас всё устраивает?
Итак, что мы видим? Доклад небезызвестного гения Г.И. Шипова, превзошедшего Эйнштейна, на обычную для него тему: "Проблема инерции в физике и т.д."
Ну, вообще-то проблема инерции в физике была решена как раз благодаря созданию общей теории относительности, но Г.И. Шипов этого очевидно не понял, как он очевидно не понял и общей теории относительности. Прежде чем что-то пытаться превзойти, неплохо это что-то хотя-бы понять. Тем, кто интересуется подробностями, очень рекомендую посмотреть моё выступление на вебинаре "Высокоэнергетические процессы в конденсированных средах" от 19.02.2025, посвящённое как раз этому вопросу: https://youtu.be/Fx6GCwi7800
Ради чего же затеян этот доклад Шипова? Судя по аннотации ради вот этого:
"Управляя частотой вращения и используя силу инерции можно заставить объект менять свою массу или двигаться в заданном направлении в соответствии с обобщенным уравнением Циолковского"
Насколько я понимаю, эта тема появилась на семинаре Климова-Зателепина в связи с открытиями самого Зателепина, поправившего второй закон Ньютона.
Ну, что сказать, если вы во всё это верите, то если вам в ближайшее время позвонят из службы безопасности вашего банка и предложат перевести деньги на безопасный счёт, вы очевидно из тех, кто будет в точности следовать всем инструкциям позвонившего.
С уважением
Андрей Чистолинов

Андрей Егоров:
Здравствуйте, коллеги, хочу выразить своё согласие с мнением Андрея Чистолинова. Также, пользуясь случаем, хочу отписаться от рассылки семинара.

YouTube (https://youtu.be/Fx6GCwi7800)
Теория торсионных полей Шипова и эффект Даннинга-Крюгера
О теории торсионных полей и единой геометризованной теории гравитации и электромагнетизма, а также о связи физики и психологии рассказывает Андрей Чистолинов

set of linear differential equations. However it has been immensely improved in the hands of relativists, in fact *the whole mathematical apparatus of non-Riemannian space is the creation of relativists during the last fifteen years* in their attempts (so far futile) to unify the differential equations of the gravitational and electromagnetic fields. It is interesting that rotating electrical machinery whose differential equations are those of a non-Riemannian space are non-holonomic dynamical systems in which also gravitational and electromagnetic energies coexist. An important connection between the *relativistic* dynamical treatment and the present *classical* dynamical treatment of the unified theory of gravitational and electromagnetic systems is that in both treatments the non-Riemannian space is subject to the restriction that the co-variant derivative of the metric tensor $g_{\alpha\beta}$ is zero.

These forms show that the equations of rotating machinery are identical with those of stationary networks if ordinary differentiation is replaced by absolute differentiation. Or in other words the appearance of the *gravitational* mass of the rotor during acceleration in a purely *electromagnetic* system changes all *ordinary* drivatives into *absolute* derivatives

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NON-RIEMANNIAN DYNAMICS OF ROTATING ELECTRICAL MACHINERY

By Gabriel Kron*

SYNOPSIS

a.) It is shown that the charges through the terminals (brushes, taps, etc.) of the innumerable types of rotating electrical machines (except of the alternator) are not true Lagrangian coordinates and *the usual form of the Equation of Motion of Lagrange*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\alpha}\right) - \frac{\partial T}{\partial x^\alpha} + \frac{\partial F}{\partial \dot{x}^\alpha} = f_\alpha$$

as interpreted by Maxwell for closed electric circuits moving in a magnetic field *is not valid for them,* neither is the Equation of Voltage (derived from the former equation by ignoring the geometrical variables.)

$$e_\alpha = R_{\alpha\beta} i^\beta + \frac{d\varphi_\alpha}{dt} = R_{\alpha\beta} i^\beta + \frac{d(L_{\alpha\beta} i^\beta)}{dt}$$

since the coordinate axes are not connected to the moving conductors.

The generalized form of the Equation of Motion valid for all electrical machinery and in general for non-holonomic dynamical systems has been given by Boltzman and Hamel as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\alpha}\right) - \frac{\partial T}{\partial x^\alpha} + \frac{\partial T}{\partial \dot{x}^\delta}\left(\frac{\partial C^\delta_k}{\partial x^n} - \frac{\partial C^\delta_n}{\partial x^k}\right) C^k_\alpha C^n_\gamma \dot{x}^\gamma + \frac{\partial F}{\partial \dot{x}^\alpha} = f_\alpha$$

which includes as a special case the usual form. The generalized form of the Equation of Voltage for rotating electrical machinery becomes

$$e_\alpha = R_{\alpha\beta} i^\beta + \frac{d\varphi_\alpha}{dt} + \psi_{t\alpha} v^t = R_{\alpha\beta} i^\beta + \frac{d(L_{\alpha\beta} i^\beta)}{dt} + L_{\gamma\delta} \frac{\partial C^\delta_k}{\partial x^t} C^k_\alpha i^\alpha v^t$$

where: 1.) the induced voltage $L_{\alpha\beta} di^\beta/dt$ is due to the variation of the *currents* 2.) the Coriolis-voltage $(dL_{\alpha\beta}/dt)i^\beta$ is due to the motion of the *coordinate axes* and 3.) the last term is a voltage due to the motion of the *conductors.*

b.) The theory of non-holonomic dynamical systems is developed in

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tensor symbolism and it is found that *the usual explicit form of the Equation of Motion for holonomic dynamical systems, that is*

$$a_{\alpha\beta} \frac{d^2x^\beta}{dt^2} + [\beta\gamma, \alpha] \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} + R_{\alpha\beta} \frac{dx^\beta}{dt} = f_\alpha$$

is also valid for "quasi-holonomic" dynamical systems (a special case of non-holonomic dynamical systems in which gravitational and electromagnetic energies coexist) provided the Christoffel symbol of Riemannian Geometry is replaced by the generalized asymmetrical Christoffel symbol of Non-Riemannian Geometry subject to the restriction that the covariant derivative of the metric tensor $a_{\alpha\beta}$ is zero. In other words it is shown that the explicit form of the *generalized Equation of Motion*

$$a_{\alpha\beta} \frac{d^2x^\beta}{dt^2} + \Gamma_{\beta\gamma,\alpha} \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} + R_{\alpha\beta} \frac{dx^\beta}{dt} = f_\alpha$$

remains invariant with respect to all quasi-holonomic transformations.

c.) Practical examples of such dynamical systems are the infinite varieties of electrical machines that have not been analyzed before from a dynamical point of view. Hence as quasi-holonomic dynamical systems the performance of all rotating electrical machinery is described during acceleration by the Equation of Motion

$$e_\alpha = R_{\alpha\beta} i^\beta + L_{\alpha\beta}\, di^\beta/dt + \Gamma_{\beta\gamma,\alpha} i^\beta i^\gamma$$

where $\Gamma_{\beta\gamma,\alpha}$ is the asymmetrical generalized Christoffel symbol and $L_{\alpha\beta}$ is the metric tensor representing all the self and mutual inductances of the windings and the moment of inertia of the rotor. The performance of any particular machine is found from that of a representative machine by a routine transformation of coordinate axes with the aid of a *transformation tensor which is nothing else but a mathematical representation of the customary connection diagram.*

By considering their connection tensors (or connection diagrams) *the various types of rotating electrical machines may also be looked upon as holonomic or non-holonomic sub-spaces of one representative non-Riemannian space.* However each non-holonomic sub-space may also be considered in its own right as a non-Riemannian space because of the existence of *electrical* non-holonomic coordinates.

When various types of rotating (or stationary) electrical apparatus are connected in any number and in any manner whatever, all the operators of the group are found simply by adding the particular operators of the individual units and transforming the sum by the transformation tensor of the group. That is *entire rotating machines are added as simple impedances.*

The Equation of Motion also can be written as

$$e_\alpha = R_{\alpha\beta}i^\beta + L_{\alpha\beta}\, di^\beta/dt + [\gamma\beta, \alpha]\, i^\gamma i^\beta + T_{\gamma\beta\alpha}i^\gamma i^\beta$$

where $[\gamma\beta, \alpha]$ is the ordinary Christoffel symbol and $T_{\gamma\beta\alpha}$ is a *skew-symmetric* tensor of rank three. Also $[\gamma\beta, \alpha]i^\gamma i^\beta$ is the Coriolis voltage and $T_{\gamma\beta\alpha}i^\gamma i^\beta$ is the rotor generated voltage and torque.

During an *infinitesimal disturbance* (hunting) the Equation of Small Oscillations of all machines is

$$de_\alpha = R_{\alpha\beta}di^\beta + L_{\alpha\beta}\frac{d(di^\beta)}{dt} + \Gamma_{\beta\gamma,\alpha}di^\beta i^\gamma + \Gamma_{\beta\gamma,\alpha}i^\beta di^\gamma + \frac{\partial\Gamma_{\beta\gamma,\alpha}}{\partial x^\delta}i^\beta i^\gamma dx^\delta.$$

d.) For the special case of *constant speed* the Equation of Motion reduces to

$$e_\alpha = Z_{\alpha\beta}i^\beta$$

where $Z_{\alpha\beta}$ is the "transient-impedance tensor." The Equation of Small Oscillation reduces to

$$de_\alpha = \bar{Z}_{\alpha\beta}di^\beta$$

where $\bar{Z}_{\alpha\beta}$ is the "motional-impedance tensor." *The routine performance calculation of the various single and interconnected machines is immensely facilitated by the use of these impedance tensors.*

The "per-unit" concept used in synchronous machines is generalized and is extended to all rotating machines by introducing the "contra-variant" voltage vector e^α, the generalized Christoffel symbol of the second kind $\Gamma^\alpha_{\beta\gamma}$ etc.

e.) In order to eliminate in the routine performance calculation of the various rotating machines the necessity of using more or less complicated formulae for the transformation of symbols to various coordinate systems, the "*tensor*" concept is introduced. In terms of tensors the Equation of Motion becomes

$$e_\alpha = R_{\alpha\beta}i^\beta + L_{\alpha\beta}\,\delta i^\beta/\delta t$$

where $\delta/\delta t$ represents "absolute" or "covariant" differentiation. The Equation of Voltage becomes

$$e_\alpha = R_{\alpha\beta}i^\beta + \delta\varphi_\alpha/\delta t.$$

These forms show that the equations of rotating machinery are identical with those of stationary networks if ordinary differentiation is replaced by absolute differentiation. Or in other words the appearance of the *gravitational* mass of the rotor during acceleration in a purely *electromagnetic* system changes all *ordinary* drivatives into *absolute* derivatives ($\partial g_{\alpha\beta}/\partial x^\gamma = 0$) and it endowes the electrical coordinates x^γ with cylindrical properties. The Equation of Small Oscillations becomes

$$\delta e_\alpha = R_{\alpha\beta}\delta i^\beta + L_{\alpha\beta}\frac{\delta(\delta i^\beta)}{\delta t} + K_{\delta\gamma\beta\alpha}i^\delta i^\beta dx^\gamma + R_{\gamma\beta\alpha}i^\beta dx^\gamma$$

where $K_{\delta\gamma\beta\alpha}$ is the generalized Riemann-Christoffel curvature tensor and $R_{\gamma\beta\alpha}$ is a tensor of rank three introduced by the resistances.

f.) These equations express the fact that *the performance calculation of rotating electrical machinery is primarily a problem in mathematical physics. It is the problem of the motion of a particle in an n-dimensional non-Riemannian (affine) space with asymmetric connection (with torsion) acted upon by a positional (non-conservative) force and opposed by a frictional force proportional to its velocity.*

g.) Throughout the paper emphasis has been laid upon a *physical* interpretation of all abstract concepts introduced from the Absolute Calculus and of the phenomena taking place inside of all machines during acceleration and hunting. Complete examples are worked out for the case of a salient-pole synchronous machine with amortisseur windings having stationary and also moving rotor coordinate axes. and for the most general unbalanced asymmetrical induction motor Scattered examples for various other machines are also given. *An attempt is also made to explain briefly all symbols new to the general reader.*

All examples worked out by the dynamical method give identical results with those found by older methods by various writers like Park, Doherty, Nickle etc. for salient-pole synchronous machines, Arnold, Dreyfus, Lyon etc. for induction and commutator machines.

The step from *complex numbers* which electrical engineers are acquainted with to *hyper-complex numbers* this paper deals with is logical and is also inevitable sooner or later as the complexity of the electrical systems to be analized increases.

INTRODUCTION

a.) At the present time the performance calculation of the various types of rotating electrical machinery is still in an unorganized state. In a *general way* it can be stated that:

1.) *Each rotating machine has a different theory* so that an engineer who can calculate the performance of say the single-phase induction motor is usually unacquainted with the performance calculation of the salient-pole alternator and vice-versa. The learning of the performance calculation of each machine requires usually an intensive study of many months.

2.) *Each leading engineer sets up his own theory* about the machine he specializes in and as a consequence the engineer who is acquainted with the method of say Steinmetz or Arnold for the calculation of the alternator must learn an *entirely new language and procedure* in studying the method of say Park or Doherty and Nickle for the same alternator, *just as if it were an entirely different machine.*

In the study of *one* particular machine the engineer has a selection among innumerable theories of innumerable writers ranging from purely analytical to purely graphical through various degrees of semi-analytical and semi-graphical methods, each theory being usually independent of the other. The diversity is still more emphasized by the various assumptions as to the nature of the design constants, etc.

In the *purely analytical* study of *one* machine the procedure usually consists of setting up a set of equations for the voltages, another set for the fluxes, a third set of equations for the torque etc. *The form of the equations, however, is different with different writers and different theories.* The analysis of all these sets of equations is long and wearysome due to the most primitive type of arithmetic that can be utilized only with the customary methods of attack.

The above statements refer mainly to the *steady-state* analysis of machines. Lately the knowledge of the *transient* performance of machines is becoming of increasing importance and the articles that already appeared indicate that the same multiplicity of equations and variety of theories for *each* machine that makes the study of the steady-state performance a drudgery will be more than ever an incentive for the engineer to follow in his studies the path of least resistance, that is to set up his own theory rather than try to find his way in a jungle adding thereby to the confusion. Curiously enough *it is the response of rotating machinery to transient disturbances that brings out most emphatically the fundamental identity (group property) of all rotating machines*, more particularly it brings out the facts that:

1) all machines have the same magnetic structure and differ only in the manner of connection of the electric circuits,

2.) in all circuits of all types of machines the same simple physical phenomenon occurs, namely $e = ir + d\varphi/dt + \psi \times$ velocity.

And in spite of the fundamental identity of all machines and of all circuit phenomena there are as many machine theories as there are types of machines and there are types of engineers.

b.) It is the purpose of this paper to point out to electrical engineers that there exists a *new* and *powerful* branch of mathematics, known under various names of Tensor-Analysis, Absolute-Calculus, Ricci-Calculus, etc., which is most ideally suited to unify the analysis of the innumerable types of rotating machines. The power of the Calculus lies in the fact that it *unifies* the analysis of a large variety of problems; in particular

1.) it reduces the analysis of the differential equations of a whole group of similar systems (like the group of all rotating electrical machines) into the analysis of *one* representative member of the group and the solutions for the other members of the group are found simply by a routine transformation of coordinates.

2.) it sets up *identical* equations for each degree of freedom of the representative system.

Although the Absolute Calculus found its first *physical* application in relativity dynamics, it is now very extensively used in problems of classical dynamics also. These classical problems mostly deal with holonomic dynamical systems with true Lagrangean coordinates and are formulated in terms of the motion of a particle in a Riemannian space. When *non-holonomic* constraints or non-holonomic transformations are introduced additional forces appear so that the particle describes paths in a Riemannian space whose geometry is called "non-holonomic." In electrical machinery the paths assume a special form and their performance can be considered as the motion of a particle in a *non-Riemannian space* with *asymmetric* connection. As far as the writer is aware the equations of a non-Riemannian space have been used in physical problems so far only in the theory of relativity, in particular in the various unified field theories.

To avoid any misunderstanding it is emphasized that the subject matter of this paper has nothing whatever to do with the theory of relativity. What relative motions are considered they all are relative motions in the classical sense treated in any book on dynamics. Also it should be emphasized that the mathematical tool itself, the Absolute Calculus, has no connection with relativity. It has been worked out by Ricci about fifty years ago when there was not even any theory of relativity. (Gauss, Riemann, etc. laid the foundation of the Calculus.) The tool is nothing else but a systematic treatment of the theory of a set of linear differential equations. However it has been immensely improved in the hands of relativists, in fact *the whole mathematical apparatus of non-Riemannian space is the creation of relativists during the last fifteen years* in their attempts (so far futile) to unify the differential equations of the gravitational and electromagnetic fields. It is interesting that rotating electrical machinery whose differential equations are those of a non-Riemannian space are non-holonomic dynamical systems in which also gravitational and electromagnetic energies coexist. An important connection between the *relativistic* dynamical treatment and the present *classical* dynamical treatment of the unified theory of gravitational and electromagnetic systems is that in both treatments the non-Riemannian space is subject to the restriction that the co-variant derivative of the metric tensor $g_{\alpha\beta}$ is zero. Also the *cylindrical* properties of the electrical coordinates are analogous to those of the recently developed five-dimensional relativity. The *cyclical* properties of the coordinates even suggest quantum-theoretical analogies. It should be noted that as an alternative to the present treatment of the performance as the motion of a *particle*, the moving *waves* inside the machines could have been also analyzed. It seems that any treatment of the non-sinusoidal space waves (subsynchronous speeds, etc.) must take over a large part of the theory of infinite-dimensional (Hilbert) spaces of modern quantum-dynamics (theory of spectrum, etc.). It is also evident that many other *formal* analogies exist between the *equations* of interconnected rotating electrical machinery and those of a group of spinning electrons.

Of course electrical machinery offer a more easily comprehensible picture to visualize the more elementary concepts of a non-Riemannian space than the unified field theory does. It is believed that the reader will find he has been familiar with most of these concepts except that they did not have such euphonious names as "metric tensor," "Christofell symbol" etc., they were only called "inductance," "generated voltage" etc. *There is nothing in this paper that can not be understood by anybody who ever attempted to understand textbooks on rotating machine performance.* The mathematical tool itself is easily learned, in words of leading authorities it has "almost miraculous" power in simplifying complicated mathematical processes and last but not least, it has a certain beauty that will positively appeal to certain types of engineers.

c.) *In the early days of rotating machinery the dynamical point of view has been successfully applied to the study of the synchronous machine* (Hopkinson etc.) by applying to it the Equation of Motion of Lagrange as has been interpreted by Maxwell for a system where geometrical and electrical (Lagrangian) coordinates are present. *But the method failed with every other machine* where the coordinate axes (brushes, connections, etc.) were not connected to the moving conductors. At that time non-holonomic dynamical systems were not analized in a systematic manner, only in the present century was their general equation set up by Boltzman and Hamel but due to its scarcity of application even today it is found only in the most advanced treatises on dynamics.

Since then sporadic attempts were made to build up a dynamical theory of rotating electrical machinery (Ingram, Dahlgren, etc.) but they all gave the same equations found by Maxwell, they all used them for the alternator only (if at all) and for no other machine. Similarly the equations derived in treatises on electrodynamics for closed electric circuits moving in a magnetic field are those of Maxwell applicable to the alternator only, although usually remarks are made that they are applicable to all dynamo-electric machines.

So Dynamics dropped out of the picture entirely and in its place electrical engineers built up an immense but loosely knit structure of physical and mathematical interpretation of the phenomena taking place *inside* the various types of machines. *The dynamical equations of Lagrange have been expressly created to avoid just such an eventuality,* they were created to predict the performance of dynamical systems by the aid of certain functions measured at the terminals, *without knowing anything about the mechanism of the phenomena inside the system.* In other words if the resistances and inductances of the various windings are given as measured at the terminals and also the diagram of connections of the terminals, no matter how complicated they are the dynamical equations can be set up and the transient and steady-state performance can be calculated in a *routine* manner by anyone who otherwise is ignorant of the various theories of rotating machines given in textbooks.

That is *this paper represents a formal mathematical approach to the performance calculation of all rotating electrical machinery and the setting up and solutions of the dynamical equations are independent of any physical theories.*

Of course dynamical equations may be interpreted physically after they are set up, but the interpretation should be considered only as an after-thought. It is employed mainly in an attempt to give a physical picture of the concepts introduced from the Absolute Calculus like "tensor" etc. and it has nothing whatever to do with the dynamical set up and their solutions. Anyone interested only in performance calculations may leave out the sections dealing with physical interpretations.

As expected, if the various terms in the generalized Equation of Motion are reinterpreted as physical quantities (flux-density, vector-potential, etc.) the new forms are nothing else but the Field Equations of Maxwell generalized for moving bodies and moving coordinate axes. It is intended to treat the various forms of the Field Equations (wave, impulse-energy, etc.) in another publication and to interpret physically the various curvature quantities ($K_{\alpha\beta}$, K) and tensors (stress-energy, electromagnetic, etc.) as they apply to rotating electrical machinery.

d.) *It has been most gratifying to find that while other methods require often many months of preparation and thought and calculation to set up, even the steady-state performance of new types of machines (and that can be done usually only by a few engineers experienced in such mental discipline) the dynamical method presented in this paper gives identical results by a routine calculation in only a few hours and it can be employed by any one who knows how to multiply matrices and find their inverse. Even in cases where other methods are practically helpless as in case of most machines with asymmetrical windings or magnetic structure and unbalanced impressed voltages or in cases where various types of machines are interconnected in any arbitrary manner, this method gives correct results with equal facility.*

Other advantages of this method are:

1.) In learning the performance calculation of one machine at the same time the performance calculation of all other machines is learned.

2.) First the transient performance of machines during acceleration is analyzed The transient performance calculation with the speed maintained constant and the steady-state performance calculation follow as special cases.

3.) *The fundamental equations derived for the transient analysis of rotating electrical machinery are identical with the fundamental equations of Dynamics and of multidimensional Differential Geometry,* hence the researches of these sciences can be applied to the study of electrical machinery.

4.) In introducing any future refinements into the analysis, for instance the effect of space-harmonics, multiple phases, slot-openings, brush-currents, etc. all fundamental equations set up in this paper remain unchanged, only the value of the constants to be substituted into the equations are to be changed.

5.) A powerful mathematical tool is acquired which can be used not only in the study of rotating machinery, which is the case with all other tools, but immediately can be applied in the most advanced studies of mathematical physics.

e.) (In a previous paper the sudden short-circuit performance with the speed maintained constant and the steady-state performance have been analyzed in greater detail from *purely physical* considerations in the vector and dyadic notation of Gibbs. In a second paper small transient variations in speed and steady hunting have been covered from the same point of view. Important labor-saving devices such as λ-matrices, complex vectors etc. have been also introduced in them.)

Anyone interested only in the *dynamical* aspect of the theory of electrical machinery may consult the following sections: II, III, IV, V, VII, VIII, IX, XXIII, XXIV, XXX.

THE REPRESENTATIVE MACHINE WITH MOVING COORDINATE AXES

I. *Hyper-complex Numbers*

a.) *The quantities the Absolute Calculus deals with are generalizations of the complex number* $A + jB$. A more compact notation for complex numbers is (A, B). A three-dimensional complex number is $iA + jB + kC$, which may also be written as (A, B, C). An evident extension is to supply each number with two indices as $Ajj + Bjk + Ckj + Dkk + \dots$. This may be denoted by the compact form

$$\begin{vmatrix} A, B, \dots \\ C, D, \dots \\ \cdot\ \cdot \\ \cdot\ \cdot \\ \cdot\ \cdot \end{vmatrix}$$

called "matrix." Set of numbers may have three or more indices.

A *scalar* is represented by a single number, a *vector* by a set of n numbers (or functions) arranged in a row, a *dyadic* by a set of n^2 numbers arranged in a square (some of them may be zero), a *triadic* by a set of n^3 numbers arranged in a cube etc. In general they will be called "*polyadics.*"

Just as in steady-state a-c theory all quantities in the equations, say Z, *stand for a complex number* $r + jx$, *similarly in the equations of this paper all quantities stand for anyone of the above hyper-complex numbers.*

b.) A component of a *vector* along axis d is denoted by an upper or lower index as A^d or A_d. One term of a *dyadic* is denoted as A_{dq},

A^{dq}, $A_d^{\cdot q}$ or $A^d_{\cdot q}$. The order or position of the indices can not be interchanged, since A_{dq} and $A_d^{\cdot q}$ belong to different sets of n^2 quantities. One term of a *triadic* is written as A_{dqf} or $A_{dq}^{\cdot\cdot f}$ etc.

An equation such as $e_\alpha = \sum_\beta Z_{\alpha\beta} i^\beta$ represents an equation along axis α where the index β assumes all the possible values of the indices representing the various axes d, q, f, etc. *There is one big difference between this notation and the usual scalar equations however. While with the usual scalar equations each coordinate axis has a different equation, in tensor notation the equations for all coordinate axes are identical.* The index α is simply replaced in turn by d, q, f, etc. for the various axes.

It should be noted that the indices to which the summation sign applies appear *twice* in the same term (once as an upper and once as a lower index). *It is an accepted convention to dispense with the summation sign* and to write the above equation as $e_\alpha = Z_{\alpha\beta} i^\beta$. If α is say d, then

$$Z_{\alpha\beta} i^\beta = Z_{dd} i^d + Z_{dq} i^q + Z_{df} i^f + \ldots.$$

The summation sign may occur twice or more in the same term. For instance $\sum_\alpha \sum_\beta R_{\alpha\beta} i^\alpha i^\beta \doteq R_{\alpha\beta} i^\alpha i^\beta$.

It should be noted that:

1.) The two indices that are written with the same letter are called "dummy" indices, the others "free" indices.

2.) Any letter may be used in the same term as a dummy index. For instance $\Gamma_{\alpha\beta\gamma} i^\beta = \Gamma_{\alpha\sigma\gamma} i^\sigma$.

3.) One of the dummy indices must be an upper, the other a lower index.

4.) Since most equations of this paper are vector equations, in every term of the equations only one of the indices will be a free index. This index must be denoted by the same letter in each term of the equation and it must be an upper or a lower index in all terms.

II. The Two Representative Machines

a.) The *most general* rotating machine consists of two structures:

1.) A *stator* having asymmetrical magnetic structure and several layers of asymmetrical windings arranged in any manner whatever.

2.) A *rotor* (not necessarily smooth) with several layers of windings arranged in any manner whatever.

It is assumed that all brushes, sliprings, taps and connections are removed from both structures.

b.) *Two representative machines will be assumed, each with different coordinate axes.*

In the first machine:

1.) Each stator winding has its own axis as a *stationary* coordinate axis.

2.) Each rotor winding has its own axis as a *moving* coordinate axis, all axes moving together with the rotor.

This machine will serve as the representative machine for which the fundamental dynamical equations will be derived.

In the second machine:

1.) The stator coordinate axes are *stationary* and are located anywhere.

2.) The rotor coordinate axes are also *stationary* and are located anywhere.

All formulae calculated for the first machine will be transformed to apply to the second machine and this second machine with stationary coordinate axes will serve as the representative machine from which the equations for all other rotating machines will be derived by a transformation of coordinate axes.

c.) *The duplication of representative machines is necessary* for the following reasons:

1.) The derivation of the fundamental equations from the Lagrangian equation is possible only for the machine with *moving* rotor coordinate axes, because *an axis must be connected to the moving conductors to serve as a true Lagrangian coordinate axis.* The only way to set the equations up for other axes is by the process of "transformation of coordinate axes."

2.) The routine calculation process of transformation for any particular machine is simplest if the representative machine has *stationary* rotor coordinate axes because all its inductances (that is all coefficients of its metric tensor) are *constants* and the troublesome cos θ and sin θ terms are eliminated in most cases.

d.) Both representative machines have an additional coordinate axis t along the shaft at right angles in space to all other axes, to represent the direction of all mechanical vectors like rotor angular velocity, angular acceleration, torque, etc.

e.) The following notation of axes will be used:

1.) Anyone of the coordinate axes of the representative machine with *moving* coordinate axes will be denoted by k, m, n ....

2.) Anyone of the axes of the machine with *stationary* coordinate axes will be denoted by $\pi\rho\sigma\tau$ ....

3.) In general the coordinate axes of *any* machine will be denoted by $\alpha, \beta, \gamma, \delta \ldots.$

Individual axes will be denoted by $q, d, b, a, t \ldots.$

III. *The Test or Design Constants*

In order to calculate the performance of any machine, it is necessary to know *two sets of quantities* either tested on an actual machine or calculated from the design.

1.) The first set consists of the *resistances* of all windings and the frictional resistance of the shaft (if any). The numbers are all constants and can be arranged along the diagonal of a square array, (matrix). It is denoted by a *symmetrical* dyadic $R_{\alpha\beta}$ (equ. 1).

$$R_{\alpha\beta} = \begin{array}{c|ccccc|} & q & d & {}^{\beta}\; b & a & t \\ \hline q & r_{qq} & 0 & 0 & 0 & 0 \\ d & 0 & r_{dd} & 0 & 0 & 0 \\ b & 0 & 0 & r_{bb} & 0 & 0 \\ a & 0 & 0 & 0 & r_{aa} & 0 \\ t & 0 & 0 & 0 & 0 & r_{tt} \\ \hline \end{array} \quad \ldots\ldots\ldots\ldots 1$$

2.) The second set of numbers represents the *self and mutual inductances* of all windings along the axes considered, and the moment of inertia of the rotor. The inductance of all rotor windings is expressed as a function of the rotor angular displacement $\theta = x^t$. The set is represented by a square array and is denoted by a *symmetrical* dyadic $L_{\alpha\beta}$. This dyadic will be called the "metric tensor" (equ. 2).

$$L_{\alpha\beta} = \begin{array}{c|ccccc|} & q & d & {}^{\beta}\; b & a & t \\ \hline {}^{\alpha}\, q & L_{qq} & L_{qd} & L_{qb}(\theta) & L_{qa}(\theta) & 0 \\ d & L_{dq} & L_{dd} & L_{db}(\theta) & L_{da}(\theta) & 0 \\ b & L_{bq}(\theta) & L_{bd}(\theta) & L_{bb}(\theta) & L_{ba}(\theta) & 0 \\ a & L_{aq}(\theta) & L_{ad}(\theta) & L_{ab}(\theta) & L_{aa}(\theta) & 0 \\ t & 0 & 0 & 0 & 0 & L_{tt} \\ \hline \end{array} \quad \ldots 2$$

The knowledge of these two sets of numbers $R_{\alpha\beta}$ and $L_{\alpha\beta}$ is sufficient to find the transient and steady-state performances of any machine assuming no magnetic saturation and no iron losses.

IV. The Ideal Representative Machine

a.) In all equations developed below the value of $L_{\alpha\beta}$ having terms with *any functions of* θ can be substituted and the performance can be found by a routine calculation. That is *a method of attack is given in these pages by which the effect of space harmonics* (slot openings, m.m.f. harmonics, etc.) *on the performance can be evaluated.*

At this point however, it is not intended to work out a complete problem on harmonics; that is reserved for a future paper. Examples will be worked out in which the assumptions as regards to the variations of the inductances with θ are the same as the most advanced treatises on machine analysis have already used (e.g. in case of the salient-pole alternator etc.) in order to show that the method of this paper gives identical results in all cases worked out by other methods with by far less mental and physical labor.

b.) An ideal rotating machine is one in which: (Fig. 1.)

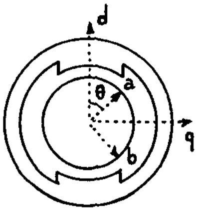


FIG. 1

1.) The rotor windings are *symmetrically* distributed around the *smooth* rotor.

2.) The stator has field-poles on it and *asymmetrical* windings anywhere.

3.) The variation of rotor self and mutual inductances with θ follows *sine curves* as shown below.

c.) In the body of the paper the following special case will be followed through:

1.) One layer of winding exists on the stator with axes along q and d and

2.) One layer of winding exists on the rotor with mutually perpendicular axes a and b at an angle θ from d and q.

(A more general example would be to assume more than two sets of axes on the rotor, for instance three sets at 120 degrees apart as it is in a three-phase synchronous machine and also additional inductances, like zero phase-sequence reactances. Since they form a special case of a general treatment of space-harmonics that is intended to be attacked systematically later, this isolated example will not be followed through here, though all equations developed are equally valid for it.)

The value of the metric tensor for this particular machine is

$$L_{mn} = \begin{array}{c|c|c|c|c|c|} m \backslash n & d_s & a & b & q_s & t \\ \hline d_s & L_{sd} & M_d \cos\theta & -M_d \sin\theta & 0 & 0 \\ \hline a & M_d \cos\theta & L_1 - L_2 \cos 2\theta & L_2 \sin 2\theta & M_q \sin\theta & 0 \\ \hline b & -M_d \sin\theta & L_2 \sin 2\theta & L_1 + L_2 \cos 2\theta & M_q \cos\theta & 0 \\ \hline q_s & 0 & M_q \sin\theta & M_q \cos\theta & L_{sq} & 0 \\ \hline t & 0 & 0 & 0 & 0 & L_{tt} \\ \hline \end{array} \quad \ldots.3$$

where $L_1 = (L_{rq} + L_{rd})/2$ and $L_2 = (L_{rq} - L_{rd})/2$. The subscript s refers to stator, r to rotor, d to direct-axis and q to quadrature-axis quantities. L represents self-inductance and M mutual inductance. L_{rd} represents the maximum value of the self-inductance of a rotor winding when its axis is along the field-poles (direct axis).

The values of R_{mn} are

$$R_{mn} = \begin{array}{c|c|c|c|c|c|} m \backslash n & q_s & b & a & d_s & t \\ \hline q_s & r_{sd} & 0 & 0 & 0 & 0 \\ \hline b & 0 & r_r & 0 & 0 & 0 \\ \hline a & 0 & 0 & r_r & 0 & 0 \\ \hline d_s & 0 & 0 & 0 & r_{sq} & 0 \\ \hline t & 0 & 0 & 0 & 0 & r_{tt} \\ \hline \end{array} \quad \ldots\ldots\ldots\ldots.4$$

d.) For the representative machine with *two* layers of windings on the stator and two on the rotor L_{mn} is given in table I. Some of the other derived constants of this machine are also given in the table to be used in the analysis of machines whose connection diagram is given in tables II and III. In the body of the paper the simpler machine will be followed through since it is also identical with the two-phase salient-pole alternator with amortisseur winding. It should be noted however that these simpler matrices in the body of the paper can be found from those given in the table simply by cancelling the rows and columns that belong to the second layers of the stator and rotor. *That is all general expressions $L_{\alpha\beta}$, $R_{\alpha\beta}$, etc. refer to those given in the table.* The simpler expressions in the body of the paper serve only as illustrations that can be easily checked.

Let $L_1^1 = (L_{rq}^1 + L_{rd}^1)/2$, $L_2^1 = (L_{rq}^1 - L_{rd}^1)/2$, $L_1^2 = L_{rq}^2 + L_{rd}^2)/2$ $L_2^2 = (L_{rq}^2 - L_{rd}^2)/2$

$L_{mn} =$ (rows m, columns n)

	d_s^2	d_s^1	a_1	a_2	b_2	b_1	q_s^1	q_s^2
d_s^2	L_{sd}^2	L_{sd}^1	$M_d \cos\theta$	$M_d \cos\theta$	$-M_d \sin\theta$	$-M_d \sin\theta$	0	0
d_s^1	L_{sd}^1	L_{sd}^1	$M_d \cos\theta$	$M_d \cos\theta$	$-M_d \sin\theta$	$-M_d \sin\theta$	0	0
a_1	$M_d \cos\theta$	$M_d \cos\theta$	$L_1^1 - L_2^1 \cos 2\theta$	$L_1^1 - L_2^1 \cos 2\theta$	$L_2^1 \sin 2\theta$	$L_2^1 \sin 2\theta$	$M_q \sin\theta$	$M_q \sin\theta$
a_2	$M_d \cos\theta$	$M_d \cos\theta$	$L_1^1 - L_2^1 \cos 2\theta$	$L_1^2 - L_2^2 \cos 2\theta$	$L_2^2 \sin 2\theta$	$L_2^2 \sin 2\theta$	$M_q \sin\theta$	$M_q \sin\theta$
b_2	$-M_d \sin\theta$	$-M_d \sin\theta$	$L_2^1 \sin 2\theta$	$L_2^2 \sin 2\theta$	$L_1^2 + L_2^2 \cos 2\theta$	$L_1^1 + L_2^1 \cos 2\theta$	$M_q \cos\theta$	$M_q \cos\theta$
b_1	$-M_d \sin\theta$	$-M_d \sin\theta$	$L_2^1 \sin 2\theta$	$L_2^1 \sin 2\theta$	$L_1^1 + L_2^1 \cos 2\theta$	$L_2^1 + L_2^1 \cos 2\theta$	$M_q \cos\theta$	$M_q \cos\theta$
q_s^1	0	0	$M_q \sin\theta$	$M_q \sin\theta$	$M_q \cos\theta$	$M_q \cos\theta$	L_{sq}^1	L_{sq}^1
q_s^2	0	0	$M_q \sin\theta$	$M_q \sin\theta$	$M_q \cos\theta$	$M_q \cos\theta$	L_{sq}^1	L_{sq}^2

$C_\pi^m =$ (rows π, columns m)

	d_s^2	d_s^1	a_1	a_2	b_2	b_1	q_s^1	q_s^2
d_s^2	1	0	0	0	0	0	0	0
d_s^1	0	1	0	0	0	0	0	0
d_r^1	0	0	$\cos\theta$	0	0	$-\sin\theta$	0	0
d_r^2	0	0	0	$\cos\theta$	$-\sin\theta$	0	0	0
q_r^2	0	0	0	$\sin\theta$	$\cos\theta$	0	0	0
q_r^1	0	0	$\sin\theta$	0	0	$\cos\theta$	0	0
q_s^1	0	0	0	0	0	0	1	0
q_s^2	0	0	0	0	0	0	0	1

$L_{\pi\sigma} =$ (rows π, columns σ)

	d_s^2	d_s^1	d_r^1	d_r^2	q_r^2	q_r^1	q_s^1	q_s^2
d_s^2	L_{sd}^2	L_{sd}^1	M_d	M_d	0	0	0	0
d_s^1	L_{sd}^1	L_{sd}^1	M_d	M_d	0	0	0	0
d_r^1	M_d	M_d	L_{rd}^1	L_{rd}^1	0	0	0	0
d_r^2	M_d	M_d	L_{rd}^1	L_{rd}^2	0	0	0	0
q_r^2	0	0	0	0	L_{rq}^2	L_{rq}^1	M_q	M_q
q_r^1	0	0	0	0	L_{rq}^1	L_{rq}^1	M_q	M_q
q_s^1	0	0	0	0	M_q	M_q	L_{sq}	L_{sq}
q_s^2	0	0	0	0	M_q	M_q	L_{sq}	L_{sq}

$\Gamma_{\pi t,\sigma} = (\Gamma_{\pi\sigma,t}) = (\Gamma_{t\pi,\sigma})$ (rows π, columns σ)

	d_r^1	d_r^2	q_r^2	q_r^1
d_s^2	0	0	$-M_d$	$-M_d$
d_s^1	0	0	$-M_d$	$-M_d$
d_r^1	0	0	$-L_{rd}^1$	$-L_{rd}^1$
d_r^2	0	0	$-L_{rd}^2$	$-L_{rd}^1$
q_r^2	L_{rq}^1	L_{rq}^2	0	0
q_r^1	L_{rq}^1	L_{rq}^1	0	0
q_s^1	M_q	M_q	0	0
q_s^2	M_q	M_q	0	0

$\Gamma_{\pi\sigma,t} =$ (rows π, columns σ)

	d_s^2	d_s^1	d_r^1	d_r^2	q_r^2	q_r^1	q_s^1	q_s^2
d_s^2	0	0	0	0	$M_d/2$	$M_d/2$	0	0
d_s^1	0	0	0	0	$M_d/2$	$M_d/2$	0	0
d_r^1	0	0	0	0	$(L_{rd}^1 - L_{rq}^1)/2$	$(L_{rd}^1 - L_{rq}^1)/2$	$-M_q/2$	$-M_q/2$
d_r^2	0	0	0	0	$(L_{rd}^2 - L_{rq}^2)/2$	$(L_{rd}^1 - L_{rq}^1)/2$	$-M_q/2$	$-M_q/2$
q_r^2	$M_d/2$	$M_d/2$	$(L_{rd}^1 - L_{rq}^1)/2$	$(L_{rd}^2 - L_{rq}^2)/2$	0	0	0	0
q_r^1	$M_d/2$	$M_d/2$	$(L_{rd}^1 - L_{rq}^1)/2$	$(L_{rd}^1 - L_{rq}^1)/2$	0	0	0	0
q_s^1	0	0	$-M_q/2$	$-M_q/2$	0	0	0	0
q_s^2	0	0	$-M_q/2$	$-M_q/2$	0	0	0	0

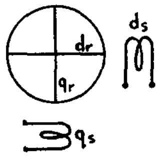
$Z_{\pi\sigma} =$ (rows π, columns σ)

	d_s^2	d_s^1	d_r^1	d_r^2	q_r^2	q_r^1	q_s^1	q_s^2
d_s^2	$r_{sd}^2 + L_{sd}^2 p$	$L_{sd}^1 p$	$M_d p$	$M_d p$	$-M_d p\theta$	$-M_d p\theta$	0	0
d_s^1	$L_{sd}^1 p$	$r_{sd}^1 + L_{sd}^1 p$	$M_d p$	$M_d p$	$-M_d p\theta$	$-M_d p\theta$	0	0
d_r^1	$M_d p$	$M_d p$	$r_r^1 + L_{rd}^1 p$	$L_{rd}^1 p$	$-L_{rd}^1 p\theta$	$-L_{rd}^1 p\theta$	0	0
d_r^2	$M_d p$	$M_d p$	$L_{rd}^1 p$	$r_r^2 + L_{rd}^2 p$	$-L_{rd}^2 p\theta$	$-L_{rd}^1 p\theta$	0	0
q_r^2	0	0	$L_{rq}^1 p\theta$	$L_{rq}^2 p\theta$	$r_r^2 + L_{rq}^2 p$	$L_{rq}^1 p$	$M_q p$	$M_q p$
q_r^1	0	0	$L_{rq}^1 p\theta$	$L_{rq}^1 p\theta$	$L_{rq}^1 p$	$r_r^1 + L_{rq}^1 p$	$M_q p$	$M_q p$
q_s^1	0	0	$M_q p\theta$	$M_q p\theta$	$M_q p$	$M_q p$	$r_{sq}^1 + L_{sq}^1 p$	$L_{sq}^1 p$
q_s^2	0	0	$M_q p\theta$	$M_q p\theta$	$M_q p$	$M_q p$	$L_{sq}^1 p$	$r_{sq}^2 + L_{sq}^2 p$

THE CONSTANTS OF THE REPRESENTATIVE MACHINES WITH TWO STATOR AND TWO ROTOR LAYERS OF WINDINGS

TABLE I.

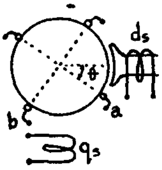
FIG. 2



$$C_\alpha^\pi = \begin{array}{c|cccc} & d_s' & d_r' & q_r' & q_s' \\ \hline d_s & 1 & 0 & 0 & 0 \\ d_r & 0 & 1 & 0 & 0 \\ q_r & 0 & 0 & 1 & 0 \\ q_s & 0 & 0 & 0 & 1 \end{array}$$

ASYMMETRICAL INDUCTION MOTOR
(SPLIT-PHASE, SINGLE-PHASE, POLYPHASE ETC.)

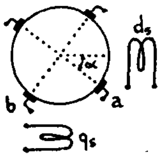
FIG. 3.



$$C_\alpha^\pi = \begin{array}{c|cccc} & d_s' & d_r' & q_r' & q_s \\ \hline d_s & 1 & 0 & 0 & 0 \\ a & 0 & \cos\theta & \sin\theta & 0 \\ b & 0 & -\sin\theta & \cos\theta & 0 \\ q_s & 0 & 0 & 0 & 1 \end{array}$$

SALIENT-POLE SYNCHRONOUS MACHINE,
SLIP-RING INDUCTION MOTOR

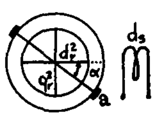
FIG. 4.



$$C_\alpha^\pi = \begin{array}{c|cccc} & d_s' & d_r' & q_r' & q_s \\ \hline d_s & 1 & 0 & 0 & 0 \\ a & 0 & \cos\alpha & \sin\alpha & 0 \\ b & 0 & -\sin\alpha & \cos\alpha & 0 \\ q_s & 0 & 0 & 0 & 1 \end{array}$$

SHUNT POLYPHASE COMMUTATOR MOTOR

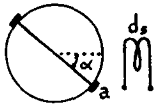
FIG. 5.



$$C_\alpha^\pi = \begin{array}{c|ccccc} & d_s' & d_r' & q_r' & d_r^2 & q_r^2 \\ \hline d_s & 1 & 0 & 0 & 0 & 0 \\ a & 0 & \cos\alpha & \sin\alpha & 0 & 0 \\ d_r^2 & 0 & 0 & 0 & 1 & 0 \\ q_r^2 & 0 & 0 & 0 & 0 & 1 \end{array}$$

SQUIRREL-CAGE REPULSION MOTOR

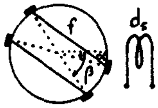
FIG. 6



$$C_\alpha^\pi = \begin{array}{c|ccc} & d_s' & d_r' & q_r' \\ \hline d_s & 1 & 0 & 0 \\ a & 0 & \cos\alpha & \sin\alpha \end{array}$$

REPULSION MOTOR

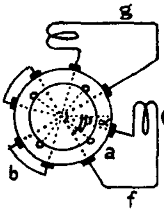
FIG. 7



$$C_\alpha^\pi = \begin{array}{c|ccc} & d_s' & d_r' & q_r' \\ \hline d_s & 1 & 0 & 0 \\ f & 0 & \cos\alpha+\cos\beta & \sin\alpha+\sin\beta \end{array}$$

DÉRI MOTOR

FIG. 8



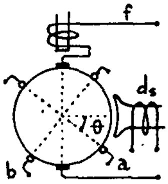
$$C_\alpha^\pi = \begin{array}{c|cccccc} & d_s' & d_r' & d_r^2 & q_r^2 & q_r' & q_s' \\ \hline f & n & \cos\beta-\cos\alpha & 0 & 0 & \sin\beta-\sin\alpha & 0 \\ a & 0 & 0 & \cos\theta & \sin\theta & 0 & 0 \\ b & 0 & 0 & -\sin\theta & \cos\theta & 0 & 0 \\ g & 0 & \sin\alpha-\sin\beta & 0 & 0 & \cos\beta-\cos\alpha & n \end{array}$$

or

$$\begin{array}{c|cccccc} & d_s' & d_r' & d_r^2 & q_r^2 & q_r' & q_s' \\ \hline f & n & \cos\beta-\cos\alpha & 0 & 0 & \sin\beta-\sin\alpha & 0 \\ d_r & 0 & 0 & 1 & 0 & 0 & 0 \\ q_r & 0 & 0 & 0 & 1 & 0 & 0 \\ g & 0 & \sin\alpha-\sin\beta & 0 & 0 & \cos\beta-\cos\alpha & n \end{array}$$

SCHRAGE MOTOR

FIG. 9.



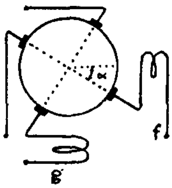
$$C_\alpha^\pi = \begin{array}{c|cccc} & d_s' & d_r' & q_r' & q_s' \\ \hline d_s & 1 & 0 & 0 & 0 \\ a & 0 & \cos\theta & \sin\theta & 0 \\ b & 0 & -\sin\theta & \cos\theta & 1 \\ f & 0 & 0 & 0 & n \end{array}$$

$$C_\alpha^\pi = \begin{array}{c|cccc} & d_s' & d_r' & q_r' & q_s' \\ \hline d_s & 1 & 0 & 0 & 0 \\ d_r & 0 & 1 & 0 & 0 \\ q_r & 0 & 0 & 1 & 1 \\ f & 0 & 0 & 0 & n \end{array}$$

SYNCHRONOUS CONVERTER WITH INTERPOLE

TABLE II.

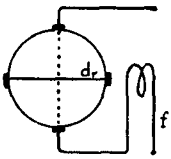
FIG. 10.



$$C_\alpha^\pi = \begin{array}{c|c|c|c|c|} & d_s' & d_r' & q_r' & q_s' \\ \hline f & n & \cos\alpha & \sin\alpha & 0 \\ \hline g & 0 & -\sin\alpha & \cos\alpha & n \\ \hline \end{array}$$

PHASE ADVANCER

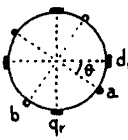
FIG. 11



$$C_\alpha^\pi = \begin{array}{c|c|c|c|} & d_s' & d_r' & q_r' \\ \hline d_r & 0 & 1 & 0 \\ \hline f & n & 0 & 1 \\ \hline \end{array}$$

COMPENSATED REPULSION MOTOR

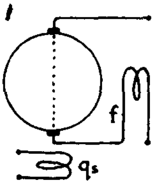
FIG. 12.



$$C_\alpha^\pi = \begin{array}{c|c|c|} & d_r' & q_r' \\ \hline d_r & 1 & 0 \\ \hline a & \cos\theta & \sin\theta \\ \hline b & -\sin\theta & \cos\theta \\ \hline q_r & 0 & 1 \\ \hline \end{array}$$

FREQUENCY CONVERTER

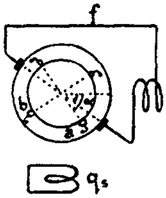
FIG. 13.



$$C_\alpha^\pi = \begin{array}{c|c|c|c|} & d_s' & d_r' & q_s' \\ \hline q_s & 0 & 0 & 1 \\ \hline f & 1 & n & 0 \\ \hline \end{array}$$

DOUBLY FED SERIES MOTOR

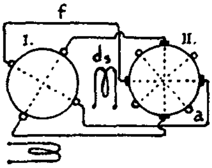
FIG. 14



$$C_\alpha^\pi = \begin{array}{c|c|c|c|c|c|c|} & d_s^1 & d_r^1 & d_r^2 & q_r^2 & q_r^1 & q_s^1 \\ \hline f & n & \cos\alpha & 0 & 0 & \sin\alpha & 0 \\ \hline a & 0 & 0 & \cos\theta & \sin\theta & 0 & 0 \\ \hline b & 0 & 0 & -\sin\theta & \cos\theta & 0 & 0 \\ \hline q_s & 0 & 0 & 0 & 0 & 0 & n \\ \hline \end{array} \text{ or } \begin{array}{c|c|c|c|c|c|c|} & d_s^1 & d_r^1 & d_r^2 & q_r^2 & q_r^1 & q_s^1 \\ \hline f & n & \cos\alpha & 0 & 0 & \sin\alpha & 0 \\ \hline d_r^2 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline q_r^2 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline a & 0 & 0 & 0 & 0 & 0 & n \\ \hline \end{array}$$

FYNN-WEICHSEL MOTOR

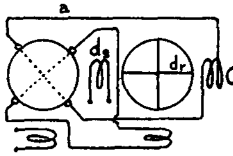
FIG. 15



$$C_\alpha^\pi = \begin{array}{c|c|c|c|c|} & d_{sI} & a_I & d_{rII} & a_{II} \\ \hline d_s & 1 & 0 & 0 & 0 \\ \hline f & 0 & 1 & 1 & 0 \\ \hline a & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

INDUCTION MOTOR AND FREQUENCY CONVERTER

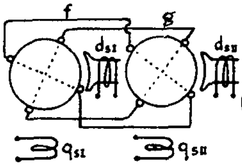
FIG. 16



$$C_\alpha^\pi = \begin{array}{c|c|c|c|c|} & d_{sI} & a_I & d_{sII} & d_{rII} \\ \hline d_s & 1 & 0 & 0 & 0 \\ \hline a & 0 & 1 & 1 & 0 \\ \hline d_r & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

INDUCTION MOTORS IN CASCADE

FIG. 17.



$$C_\alpha^\pi = \begin{array}{c|c|c|c|c|c|c|c|c|} & d_{rI} & d_{rII} & q_{rII} & q_{rI} & d_{sI} & q_{sI} & d_{sII} & q_{sII} \\ \hline d_r & 1 & \cos\delta & -\sin\delta & 0 & 0 & 0 & 0 & 0 \\ \hline q_r & 0 & \sin\delta & \cos\delta & 1 & 0 & 0 & 0 & 0 \\ \hline d_{sI} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline q_{sI} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline d_{sII} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline q_{sII} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array} \text{ or } \begin{array}{c|c|c|c|c|c|c|c|c|} & a_I & b_I & a_{II} & b_{II} & d_{sI} & q_{sI} & d_{sII} & q_{sII} \\ \hline f & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline g & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline d_{sI} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline q_{sI} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline d_{sII} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline q_{sII} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

TWO SYNCHRONOUS MACHINES

TABLE III.

V. The Equation of Motion

a.) Let in the representative machine with *moving coordinate axes* the following quantities be defined:

1.) the total number of *charges* that passed through any winding, counted from a definite time and the instantaneous angular displacement of the rotor, measured from the center of the field-pole are denoted by x^k where k may have any value $q, d, \ldots t$. (x^t is also denoted as θ).

2.) the value of the instantaneous *current* in each winding and the instantaneous angular velocity of the rotor are $i^k = dx^k/dt$

3.) the instantaneous terminal *voltage* applied to any winding and the instantaneous applied shaft torque are e_k.

The instantaneous stored *kinetic energy* is $T = (1/2)L_{mn}i^m i^n$ consisting of the sum of the magnetic and the mechanical kinetic energy. The instantaneous *dissipation function* (one-half of the dissipated power) is $F = (1/2)R_{mn}i^m i^n$. The *potential energy* of the machine is zero.

In a general dynamical system several types of coordinates may exist, geometrical, elastic, magnetic, etc. *In rotating electrical machinery two types of coordinates occur, one geometrical coordinate θ and $n - 1$ electrical coordinates (charges).* Since electrical coordinates differ in many respects from geometrical ones the resultant dynamical equations are special cases of the usual equations of Particle Dynamics allowing special treatments. (Electrical coordinates are also called "cyclic" coordinates.)

b.) *The Equation of Motion of Lagrange for a system containing kinetic energy and dissipation is*

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^k}\right) - \frac{\partial T}{\partial x^k} + \frac{\partial F}{\partial \dot{x}^k} = e_k} \quad\ldots\ldots\ldots\ldots\ldots\ldots 5$$

Substituting the values of T and F

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\partial L_{mn}i^m i^n}{\partial i^k}\right) - \frac{1}{2}\frac{\partial L_{mn}i^m i^n}{\partial x} + \frac{1}{2}\frac{\partial R_{mn}i^m i^n}{\partial i^k} = e_k.$$

But

$$\frac{1}{2}\frac{\partial L_{mn}i^m i^n}{\partial i^k} = L_{mk}i^m$$

and

$$\frac{1}{2}\frac{\partial R_{mn}i^m i^n}{\partial i^k} = R_{mk}i^m$$

Also

$$\frac{d(L_{mk}i^m)}{dt} = L_{mk}\frac{di^m}{dt} + \frac{dL_{mk}}{dt}i^m = L_{mk}\frac{di^m}{dt} + \frac{\partial L_{mk}}{\partial x^n}\frac{dx^n}{dt}i^m$$

$$= L_{mk}\frac{di^m}{dt} + \frac{\partial L_{mk}}{\partial x^n}i^m i^n.$$

Substituting

$$e_k = R_{mk}i^m + L_{mk}\frac{di^m}{dt} + \left(\frac{\partial L_{mk}}{\partial x^n} - \frac{1}{2}\frac{\partial L_{mn}}{\partial x^k}\right)i^m i^n \ldots\ldots\ldots 6$$

In this equation it is customary to divide the first term in parenthesis into two components by interchanging the indices m and n, so that

$$\frac{\partial L_{mk}}{\partial x^n}i^m i^n = \frac{1}{2}\left(\frac{\partial L_{mk}}{\partial x^n} + \frac{\partial L_{nk}}{\partial x^m}\right)i^m i^n \ldots\ldots\ldots\ldots\ldots 7$$

The expression in the parenthesis of equ. 6 is a triadic, it will be denoted by $[mn, k]$ and called the "Christoffel symbol of the first kind"

$$\boxed{[m\,n, k] = \frac{1}{2}\left(\frac{\partial L_{mk}}{\partial x^n} + \frac{\partial L_{nk}}{\partial x^m} - \frac{\partial L_{mn}}{\partial x^k}\right)} \ldots\ldots\ldots\ldots 8$$

Hence the Equation of Motion of the representative machine is

$$e_k = R_{mk}i^m + L_{mk}\frac{di^m}{dt} + [mn, k]\, i^m i^n \ldots\ldots\ldots\ldots 9$$

c.) *It will be assumed that each term of this equation will be represented by the same symbols in any coordinate system,* that is the Equation of Motion will be assumed to be represented in any coordinate system by replacing $[mn, k]$ by $\Gamma_{\beta\gamma,\,\alpha}$ representing any function of $L_{\alpha\beta}$

$$\boxed{e_\alpha = R_{\alpha\beta}\, i^\beta + L_{\alpha\beta}\frac{di^\beta}{dt} + \Gamma_{\beta\gamma,\,\alpha}\, i^\beta i^\gamma} \ldots\ldots\ldots\ldots 10$$

VI. Calculation of $\Gamma_{mn,\,k}$

a.) It will be found advantageous to define $\Gamma_{mn,\,k}$ for the representative machine just as it follows from the dynamical equation 6

$$\Gamma_{mn,\,k} = \frac{\partial L_{mk}}{\partial x^n} - \frac{1}{2}\frac{\partial L_{mn}}{\partial x^k} \ldots\ldots\ldots\ldots\ldots\ldots 11$$

instead of as in equ. 8. *Both definitions of $\Gamma_{mn,\,k}$ will give identical voltages and torques in all coordinate systems* but with the latter definition the actual calculation of voltages is cut into two and the physical definitions are expressed in simpler forms

b.) In calculating $\Gamma_{mn,\,k}$ from equ. 11 it should be noted that one of the indices must be t since L_{mn} is a function of $x^t = \theta$ only. That is from equ. 11

$$1.)\ \text{if } n = t \quad \Gamma_{mt,\,k} = \frac{\partial L_{mk}}{\partial x^t} \qquad\qquad 12$$

$$2.)\ \text{if } k = t \quad \Gamma_{mn,\,t} = -\frac{1}{2}\frac{\partial L_{mn}}{\partial x^t} \qquad\qquad 13$$

$$3.)\ \text{if } m = t \quad \Gamma_{tn,\,k} = 0 \qquad\qquad 14$$

Differentiating every term of L_{mn} as given in equ. 3 with respect to θ then

$$\Gamma_{mt,k} = \begin{array}{c|c|c|c|c|} & d_s & a & b & q_s \\ \hline d_s & 0 & -M_d \sin\theta & -M_d \cos\theta & 0 \\ \hline a & -M_d \sin\theta & 2(L_{rq} - L_{rd}) \sin\theta \cos\theta & (L_{rq} - L_{rd})(\cos^2\theta - \sin^2\theta) & M_q \cos\theta \\ \hline b & -M_d \cos\theta & (L_{rq} - L_{rd})(\cos^2\theta - \sin^2\theta) & -2(L_{rq} - L_{rd}) \sin\theta \cos\theta & -M_q \sin\theta \\ \hline q_s & 0 & M_q \cos\theta & -M_q \sin\theta & 0 \\ \hline \end{array} \quad ..15$$

$$\Gamma_{mn,t} = \begin{array}{c|c|c|c|c|} & d_s & a & b & q_s \\ \hline d_s & 0 & (M_d \sin\theta)/2 & (M_d \cos\theta)/2 & 0 \\ \hline a & (M_d \sin\theta)/2 & (L_{rd} - L_{rq}) \sin\theta \cos\theta & (L_{rd} - L_{rq})(\cos^2\theta - \sin^2\theta)/2 & -(M_q \cos\theta)/2 \\ \hline b & (M_d \cos\theta)/2 & (L_{rd} - L_{rq})(\cos^2\theta - \sin^2\theta)/2 & (L_{rq} - L_{rd}) \sin\theta \cos\theta & (M_q \sin\theta)/2 \\ \hline q_s & 0 & -(M_q \cos\theta)/2 & (M_q \sin\theta)/2 & 0 \\ \hline \end{array} \quad ..16$$

For instance $\Gamma_{bt,\, qs} = -\, M_d \sin\theta$, $\Gamma_{ds\; qs,\, t} = 0$.

c.) *With the aid of the two divisions of* $\Gamma_{mn,\,k}$ *into* $\Gamma_{mt,\,k}$ *and* $\Gamma_{mn,\,t}$ *the Equation of Motion can be divided into two components.* One is the *Equation of Voltage* found by allowing the free index assume any value but t

$$e_k = R_{mk}i^m + L_{mk}\frac{di^m}{dt} + \Gamma_{mt,\,k}i^m i^t \quad (k = d_s, a, b, q_s)\ldots.17$$

representing the voltage equation of each coordinate axis. The other component is the *Equation of Torque* found by allowing the free index assume only the value t

$$e_t = R_{tt}i^t + L_{tt}\frac{di^t}{dt} + \Gamma_{mn,\,t}i^m i^n \ldots\ldots\ldots\ldots\ldots.18$$

where the torque developed by the machine is

$$\boxed{\text{Torque} = \Gamma_{mn,\,t}i^m i^n}\ldots\ldots\ldots\ldots\ldots\ldots.19$$

For instance if $k = q_s$ than the instantaneous voltage in the stator along axis q_s is found from equ. 17

$$e_{qs} = R_{qs\;qs}i^{qs} + M_q \sin\theta\frac{di^a}{dt} + M_q \cos\theta\frac{di^b}{dt} + i^a M_q \cos\theta i^t - i^b M_q \sin\theta i^t.$$

d.) If the labor-saving device is not followed and $\Gamma_{mn,\,k}$ is defined as the sum of *three* components as in equ. 8 than $\Gamma'_{mn,\,k}$ is expressed in three matrices

$$1.)\ \Gamma'_{mt,k} = \frac{1}{2}\frac{\partial L_{mk}}{\partial x^n} = \text{one half of equ. } 15\ldots\ldots\ldots\ldots20$$

$$2.)\ \Gamma'_{tn,k} = \frac{1}{2}\frac{\partial L_{nk}}{\partial x^m} = \text{one half of equ. } 15\ldots\ldots\ldots\ldots21$$

$$3.)\ \Gamma'_{mn,t} = -\frac{1}{2}\frac{\partial L_{mn}}{\partial x^t} = \text{equ. } 16\ldots\ldots\ldots\ldots\ldots\ldots22$$

With this definition the Equation of Voltage is

$$e_k = R_{mk}i^m + L_{mk}\frac{di^m}{dt} + \Gamma_{(mt),\,k}\, i^m i^t \ldots\ldots\ldots\ldots23$$

where

$$\Gamma_{(mt),\,k} = \Gamma_{(tm),\,k} = \Gamma_{mt,\,k} + \Gamma_{tm,\,k} \ldots\ldots\ldots\ldots\ldots 24$$

The matrix of $\Gamma_{(mt),\,k}$ is equ. 15, that is the same as that of $\Gamma_{mt,\,k}$.

e.) Also it should be noted that equ. 11 could have been defined as $\Gamma_{nm,\,k}$ instead of $\Gamma_{mn,\,k}$. With this definition equ. 15 would have been denoted as $\Gamma_{tm,\,k}$ instead of $\Gamma_{mt,\,k}$.

THE REPRESENTATIVE MACHINE WITH STATIONARY COORDINATE AXES

VII. The Transformation of Coordinate Axes

a.) Let a new representative machine be assumed in which the rotor coordinate axes are *stationary* along the direct and quadrature axes d_r and q_r. Then *the old variables x^a and x^b are replaced by new variables x^{dr} and x^{qr}*, while the variables x^{ds}, x^{qs} and x^t remain unchanged.

In order to set up a relation between the new and the old *variables* if possible, the variation of the mutual inductance between the new and the old coordinate axes will be examined.

Returning to equ. 3 the mutual inductance between axes a and d_s is $M_d \cos\theta$ where M_d is the value of the mutual inductance when axis a is opposite axis d_s. Hence the mutual inductance between axes a and d_r is assumed as $L_{rd}\cos\theta$ where L_{rd} is the self-inductance of axis a when it coincides with axis d_r.

Similarly the mutual inductance between axis b and d_r is assumed as $-\ L_{rd}\sin\theta$.

b.) Hence the following identity exists between the *flux-linkages* set up at axis d_r by currents at the new and by currents at the old coordinate axes

$$i^{dr}L_{rd} = i^a L_{rd}\cos\theta - i^b L_{rd}\sin\theta$$

since i^{qr} produces no flux-linkages at axis d_r.

The relation between the *currents* is found by cancelling L_{rd}

$$i^{dr} = i^a\cos\theta - i^b\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots 25$$

Following similar reasonings with respect to the flux-linkages set up at axis q_r by currents at the new and at the old coordinate axes

$$i^{qr} = i^a\sin\theta + i^b\cos\theta$$

The relations between the *differentials* of the new variables dx^r and the old variables dx^m are found from the two current-equations by cancelling dt

$$dx^{dr} = \cos\theta\, dx^a - \sin\theta\, dx^b \quad \ldots\ldots 26$$

$$dx^{qr} = \sin\theta\, dx^a + \cos\theta\, dx^b \quad \ldots\ldots 27$$

c.) *The coefficients of the differentials can be represented in a square matrix called the "transformation tensor" and denoted by* C^π_m. Since i^{qs}, i^{qr} and i^t remain unchanged $C^{qs}_{qs} = 1$ etc. That is the transformation tensor C^π_m is

$$C^\pi_m = \begin{array}{c|c|c|c|c|c|} & d_s & d_r & {}^\pi\; q_r & q_s & t \\ \hline {}_m d_s & 1 & 0 & 0 & 0 & 0 \\ a & 0 & \cos\theta & \sin\theta & 0 & 0 \\ b & 0 & -\sin\theta & \cos\theta & 0 & 0 \\ q_s & 0 & 0 & 0 & 1 & 0 \\ t & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array} \quad \ldots\ldots 28$$

(Since the determinant is unity, the transformation is called "orthogonal.") Hence

$$\boxed{dx^\pi = C^\pi_m\, dx^m} \quad \ldots\ldots 29$$

Also

$$\boxed{i^\pi = C^\pi_m\, i^m} \quad \ldots\ldots 30$$

As an example the current along the new axis dr is found from its values along the old axes

$$i^{dr} = C^{dr}_{ds} i^{ds} + C^{dr}_{a} i^a + C^{dr}_{b} i^b + C^{dr}_{qs} i^{qs} = \cos\theta\, i^a - \sin\theta\, i^b.$$

The values of the other vectors and polyadics along the *new* coordinate axes are also found with the aid of the transformation tensor C^π_m from their values along the old coordinate axes, but their formulae of transformation are different from that of i^π in equ. 30.

VIII. Quasi-holonomic Transformations

a.) *A very important point is to note that while an equation was set up between the differentials of the old and the new variables, no equation can* *be set up between the variables themselves* such as $x^{dr} = f^{dr}(x^a, x^b \ldots)$. Consequently equ. 26 can not be written as

$$dx^{dr} = \frac{\partial f^{dr}}{\partial x^a} dx^a + \frac{\partial f^{dr}}{\partial x^b} dx^b \ldots\ldots\ldots\ldots\ldots\ldots 31$$

If this last equation is assumed then

$$\frac{\partial f^{dr}}{\partial x^a} = C_a^{dr} = \cos\theta \ldots\ldots\ldots\ldots\ldots\ldots 32$$

and the following contradiction results

$$\frac{\partial C_a^{dr}}{\partial x^t} = -\sin\theta = \frac{\partial^2 f^{dr}}{\partial x^a \partial x^t} = \frac{\partial^2 f^{dr}}{\partial x^t \partial x^a} = \frac{\partial C_t^{dr}}{\partial x^a} = 0.$$

Hence C_a^{dr} *can not be written as* $\partial f^{dr}/\partial x^a$, *in the expression* $\partial C_m^\pi/\partial x^n$ *the indices m and n can not be interchanged,* the differential equation 26 can not be integrated and the function $f(x^{dr}, x^a \ldots) = 0$ does not exist. For this reason dx^{dr}, dx^{qr} *are not exact differentials* and consequently the treatment of rotating electrical machinery differs from that of other dynamical systems with true Lagrangian coordinates (called "holonomic" dynamical systems) where always an equation can be set up between the old and the new *variables* themselves.

b.) *However the treatment of electrical machinery differs also from that of a general non-holonomic dynamical system due to the presence of both geometrical and electrical coordinates which influences the forms of the transformation tensor and of the metric tensor the following way:*

1.) The coefficients of the transformation tensor equ. 28 are either constants or are functions of $x^t = \theta$ only. (In general they are functions of all the old coordinates.)

2.) The coordinate x^t *always* remains unchanged, that is x^t can be considered as an old and also as a new variable. Some of the other coordinates also remain unchanged.

3.) The metric tensor $L_{\alpha\beta}$ is a function of x^t only, similarly $\Gamma_{\alpha\beta,\gamma}$. (In general they are functions of all the old variables.)

Due to this special form the number of unknowns (dependant variables) in the transformed equations is n, that is the same as the number of dynamical equations. Before the transformation the variables are x^t and the $n - 1$ currents dx^m/dt (or the $n - 1$ electrical coordinates x^m) after the transformation the variables are x^t and the $n - 1$ new currents dx^π/dt or x^π. (*In case of general non-holonomic dynamical systems* *the number of variables in the n transformed dynamical equations is 2n.* They are the n old coordinates occurring in $L_{\alpha\beta}$ and C^m_π and the n new differentials or velocities. The n equations of transformations supply the additional equations.)

Since in the transformed equations the new variables can be calculated, the expression "non-holonomic coordinates" is justified, even though in case of general non-integrable transformations the expression is meaningless and only "differentials of non-holonomic coordinates" is admissible. In other words "electrical non-holonomic coordinates" do exist but "geometrical non-holonomic coordinates" do not.

Following a suggestion of Lorentz who calls the system of electric charges moving in a ponderable body a "quasi-holonomic" system (although being non-holonomic it is solved as if it were holonomic) *rotating electrical machinery and similar dynamical systems will be called "quasi-holonomic" dynamical systems* since they are soluble as if they were holonomic dynamical systems.

c.) With respect to the electrical coordinates the matrix of transformation appears as containing only constant terms since θ is not transformed. Hence the inverse transformation tensor C^m_π is found simply by calculating the inverse of the matrix of C^π_m. (In the general case where $C^\pi_m = \partial x^\pi / \partial x^m$ first the old coordinates must be expressed in terms of the new coordinates and then only can C^m_π be calculated by differentiation.) The inverse transformation tensor is

$$C^m_\pi = \begin{array}{c|c|c|c|c|c|} & d_s & a \quad {}^m & b & q_s & t \\ \hline d_s & 1 & 0 & 0 & 0 & 0 \\ \hline d_r & 0 & \cos\theta & -\sin\theta & 0 & 0 \\ \hline q_r & 0 & \sin\theta & \cos\theta & 0 & 0 \\ \hline q_s & 0 & 0 & 0 & 1 & 0 \\ \hline t & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array} \quad \ldots\ldots\ldots 33$$

IX. The Connection Tensor of Machines

a.) Just as the representative machine with stationary coordinate axes differs from that with moving coordinate axes only by a transformation or connection tensor by the aid of which all polyadics L_{mn}, $\Gamma_{mn,k}$ etc. are transformed into $L_{\pi\sigma}$, $\Gamma_{\pi\sigma,\tau}$ etc. according to certain rules developed later, similarly *all* other *rotating machines differ from the representative machine with stationary coordinate axes by a transformation tensor* C^α_π. Each machine has its own transformation or *connection tensor, which in fact is nothing else but a mathematical representation of its connection diagram.*

If the connection diagram of the machine is given, its connection tensor can be set up immediately in a few minutes and that found all its operators $\Gamma_{\pi\sigma,\tau}$ etc. and hence *its performance can be found by only a routine calculation*, which may be long or short depending on the complexity of the connection diagram, but *without any further analysis* or research into the physics of the problem. In particular:

1.) the steady-state performance of all single and interconnected machines and the steady hunting performance are found by arithmetical calculations.

2.) in case of machines with stationary coordinate axes (to which by far the largest number of machines belong or *can be reduced to*) the performance calculation for sudden short-circuits with constant speed maintained, and for transient speed variations superimposed upon a steady state is automatically put into a form in which the Expansion Theorem of Heaviside (which is used mostly in *stationary* net work analysis) can be immediately applied *without the use of operational transformations.*

In the accompanying table the connection diagrams and connection tensors of a few representative types of machines used in industry are given. They were discussed in detail in the previous papers where important labor-saving methods (complex vectors, λ-matrices, etc.) were also introduced.

It should be noted that:

1.) When two circuits f and g are in series they can be replaced by h and i^f becomes i^h, also i^g becomes i^h

2.) In machines with sliprings on a rotor circuit like the Schrage motor, (Fig. 8) [or the synchronous converter (Fig. 9)] where sliprings are connected to the *second* rotor layer, the moving coordinate axes of the sliprings a_{r2}, b_{r2} can be replaced by the stationary coordinate axes d_{r2} and q_{r2} in practically all problems.

3.) n represents the ratio of turns.

b.) The power of the new tool can be appreciated especially when analyzing a *group of machines* interconnected in any manner whatever as represented by a tensor C_α^π. *Every operator $L_{\alpha\beta}$, $\Gamma_{\alpha\beta,\gamma}$ of the group is equal to the sum of that particular operator of each individual unit,* transformed by C_α^π. The group connection diagram and the group connection tensor of a few interconnected machines are given in figs. 15, 16 and 17.

In case of balanced polyphase machines with balanced impressed voltages only one phase may be studied. To do that all "q" indices along the stator and the brushes may be replaced by a "d" index and the corresponding terms multiplied by $-j$ representing a fundamental frequency time lag. Also all slip-ring indices "b" may be replaced by an "a" index and the corresponding term multiplied by $-k$ representing a slip-frequency time lag.

c.) *Rotating machines can be grouped according to the form of the matrix of the connection tensor into three classes:*

1.) The matrix of C^{π}_{α} is *square*, that is the number of old coordinates is equal to the number of new coordinates.

2.) The number of new coordinate axes is less and the matrix of C^{π}_{α} is *rectangular* (also called "singular matrix").

3.) The number of new coordinate axes is more.

In the two latter cases the determinant of the transformation matrix is zero and the inverse of C^{π}_{α} can not be defined without further assumptions.

Rotating machines also can be grouped into two classes according to the nature of the variables occurring in the transformation tensor:

1.) Those connection tensors that contain no θ can be represented as $\partial x^{\pi}/\partial x^{\alpha} = C^{\pi}_{\alpha}$.

2.) Those that contain θ again represent non-integrable transformations in passing from the representative machine with stationary coordinate axes to the particular machine and C^{π}_{α} can not be represented as $\partial f^{\pi}/\partial x^{\alpha} = \partial x^{\pi}/\partial x^{\alpha}$.

d.) It should also be noted that from fig. 2 the connection tensor of the salient-pole alternator with amortisseur windings, the asymmetrical unbalanced induction motor (split-phase, capacitor, etc. motors) the single-phase induction motor and multiple-cage induction motor is the unit matrix (having unity in the main diagonal and zero everywhere). The final transient and steady-state formulae of these machines are exactly identical in using the method of the paper, the difference is only in the type of terminal voltage (*d–c.*, *a–c.*, balanced, unbalanced etc.) that is applied to them.

X. Sinusoidal Space Waves

a.) The physical concepts so far introduced are:

1.) the charges and angular displacement x^{α}

2.) the currents and angular velocity i^{α}

3.) the applied terminal voltages and torque e_{α}

4.) the inductances and moment of inertia $L_{\alpha\beta}$
5.) the resistance and friction $R_{\alpha\beta}$
6.) the flux-linkages and angular momentum $L_{\alpha\beta}i^{\beta}$.

Besides these concepts no other physical quantity need to be introduced throughout the rest of the paper. All other expressions that occur in the equations like $\Gamma_{\alpha\beta,\gamma}i^{\alpha}i^{\beta}$ or $L_{\alpha\beta}di^{\beta}/dt$ etc., can be considered simply as mathematical symbols. This point of view has been followed more or less consistently by other writers. In this paper however it is intended to interpret physically every symbol and expression that occurs in the equations to facilitate the understanding of the meaning of the symbols, their peculiarities, manner of transformation etc. In fact rotating electrical machinery offer a most clear-cut, easily comprehensible physical picture by which the otherwise abstract concepts of the Absolute Calculus can be immediately visualized.

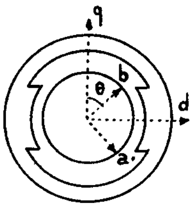


FIG. 18

It will be assumed for the sake of a physical interpretation of symbols, that at any one instant during acceleration in each layer of winding the current-density and flux-density waves are sinusoidally distributed in space.

These assumptions were made in the previous papers in setting up the fundamental equations and the equations derived there are identical with those derived here, though *in this paper the assumptions are only the sinusoidal variation of inductances.* Hence the assumptions are legitimate; besides the actual space variation of quantities in most machines is not far from sinusoidal, especially in the rotor. In any case it is only assumed for visualization that the machine with sinusoidal variation of inductances is replaced by another machine with sinusoidal space variation of current and flux-densities, both machines giving identical results.

[b.) The current i^{ds} is the current in the winding surrounding the field pole along axis d_s. This current however becomes $i^{qs} = i.q_s$ when considered as a current-density wave. That is what here is called i^{ds} it was called in the previous paper $i.q_s$, i^{qs} was called $i.d_s$ etc. However all results of both points of view are identical if:

1.) the unit vectors and indices d and q are interchanged, also a and b

2.) the instantaneous position of the coordinate axes used in the previous paper are given in Fig. 18, while their position as used in this paper is given in Fig. 1.]

XI. The Physical Interpretation of Symbols

a.) A *two-pole* sinusoidal space wave in a winding like current density, can be represented by a vector drawn from the center of the rotor toward the positive maximum value of the wave.

An n-dimensional space wave or vector consists of two-dimensional waves or vectors in each layer of winding, with an additional dimension along the axis of the rotor.

b.) In every rotating machine the following *n-dimensional* space waves or vectors can be assumed to exist at any one instant during acceleration:

1.) a current-density and velocity vector i^α

2.) a vector $L_{\alpha\beta}i^\beta$ which includes the flux-linkage vector φ_α (magnetic vector potential) and the angular momentum $L_{tt}i^t$

3.) the rotor flux-density vector $\Gamma_{\alpha t,\,\gamma}i^\alpha = \bar{\Psi}_{t\gamma}$ (or $\Gamma_{(\alpha t),\,\gamma}i^\alpha$). Although the expression stands for a dyadic, still the dyadic can be represented physically by a space vector. This calls attention to the fact that two kinds of physical vectors can be differentiated "polar" and "axial" vectors. The polar vectors like the above two are mathematically represented by a vector φ_α while an axial vector, like the flux density is mathematically represented by a dyadic $\Psi_{t\gamma}$.

In addition the following voltage vectors exist:

1.) the impressed voltage vector and shaft torque e_α

2.) the resistance-drop vector and frictional drop $R_{\alpha\beta}i^\beta$

3.) the vector $L_{\alpha\beta}di^\beta/dt$ which includes the *induced* voltage vector $d\bar{\varphi}_\alpha/dt$ and the time rate of change of angular momentum $L_{tt}di^t/dt$

4.) the vector $\Gamma_{\alpha\beta,\,\gamma}i^\alpha i^\beta$ which includes the *generated* voltage vector $\bar{\Psi}_{t\gamma}i^t$ and the torque developed by the machine $\Psi_{\beta t}i^\beta$. The two flux density vectors $\bar{\Psi}_{t\alpha}$ and $\Psi_{\alpha t}$ are not identical. (See section XXVI.)

c.) In terms of these space vectors the *Equation of Voltage* (equ. 17) can be written as

$$e_\alpha = R_{\alpha\beta}i^\beta + \frac{d\bar{\varphi}_\alpha}{dt} + \bar{\Psi}_{t\alpha}i^t \quad \ldots\ldots\ldots\ldots\ldots\ldots 34$$

if $d\varphi_\alpha/dt$ is defined as $L_{\alpha\beta}di^\beta/dt$, and the Equation of Torque (equ. 18) as

$$e_t = R_{tt}i^t + L_{tt}\frac{di^t}{dt} + \Psi_{\alpha t}i^\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots 35$$

These equations are the Field Equations of Maxwell generalized for moving bodies, expressed in terms of the magnetic vector potential (assuming no electrostatic field).

d.) *Hence in each layer of winding at all instant maximum four two-dimensional vectors can be drawn in space always forming a closed polygon:*

1.) the impressed voltage vector, if any,
2.) the resistance drop vector,
3.) the induced voltage vector,
4.) the generated voltage vector, if any.

e.) [In a former publication the writer developed a *purely graphical* steady-state analysis of rotating machinery in which the loci of all currents, having circular, elliptical or any other shape, were found with the aid of *these four vectors only* and the current vector, which all do exist actually inside the machine in each layer (that is they are measurable). The advantages of this purely physical, graphical theory compared with other graphical theories are:

1.) the only formula used in the construction of loci is r/x which only serves as scale of measurements

2.) beside these actually existing vectors, practically no other auxiliary construction lines are used

3.) at any desired point of any locus the magnitude of all currents, fluxes and voltages actually existing in each layer of winding can be immediately constructed without any auxiliary lines, in their correct space and time phase relation to each other, just as they exist inside the machine.

The results of the graphical treatment are identical with those of this paper.

It seems that any eventual purely *graphical, transient* theory of rotating machinery should be built upon the analytically and physically sound principle of representing only *actually existing* vectors on the diagram instead of hypothetical leakage fluxes, magnetizing currents, etc. or even any other auxiliary construction lines.]

f.) In connection with the space vectors defined above a physical definition can be given to the polyadics of this paper as follows:

1.) a dyadic is an *operator* which moves a space vector from one part of space into another part by rotations and extensions. For instance $L_{\alpha\beta}$ changes the current vector i^β into a flux linkage vector $\varphi_\alpha = L_{\alpha\beta}i^\beta$. In each layer of winding the position of φ_α is different from that of i^β

2.) a triadic is an *operator* which changes *two* vectors into one vector. That is $\Gamma_{\alpha t,\ \beta}$ changes the current vector i^α and the velocity vector i^t into a generated voltage vector $e_\beta = \Gamma_{\alpha t,\ \beta} i^\alpha i^t$. Or $\Gamma_{\alpha\beta,\ t}$ changes two current vectors i^α and i^β into a torque vector $e^t = \Gamma_{\alpha\beta,\ t} i^\alpha i^\beta$.

THE THEORY OF TENSORS

XII. Polyadics and Tensors

One of the aims of the Absolute Calculus is to investigate how a particular polyadic behaves when a new coordinate system is introduced.

Let a *physical space vector*, say a generated voltage vector, be considered, whose components A^a and A^b are known along the moving coordinate axes b and a. If stationary coordinate axes d^r and q^r are introduced and the *same* physical vector is constructed again from its components A^{dr} and A^{qr} along the new axes, then *often the resultant vector does not coincide with its former position.* The vector may be even zero in the new coordinate system, although everything remained the same except the axes from which measurements are made.

If the magnitude and position of a vector is the same in any coordinate system, it is called a "tensor" of rank one. Similarly if the result of an operation by a polyadic is the same vector even if the polyadic is expressed in various coordinate systems, the polyadic is called a "tensor" (or "invariant").

A tensor is transformed by multiplying each of its indices by the transformation tensor C^β_α. For instance the *new* coefficients $A_{\alpha\beta\gamma}\ldots$ of a tensor are found from the *old* coefficients $A_{mnk}\ldots$ by

$$\boxed{A_{\alpha\beta\gamma\ldots} = A_{mnk\ldots} C^m_\alpha C^n_\beta C^k_\gamma \ldots} \qquad 36$$

A polyadic which is not a tensor is transformed into a new coordinate system by a more complicated formula. A great part of the work below will consist of determining which polyadics are tensors and of finding the formulae of transformation of those that are not tensors.

So far the only tensors established are dx^π and i^π since their formulae of transformation as given in equations 20 and 22 are those of tensors.

If the matrix of transformation is not square but rectangular all equations developed below and all physical interpretations are also valid for them, except the above physical definition of a tensor. *The* *new polyadics derived with the aid of the rectangular matrices are called "induced" polyadics*

XIII. Covariant Tensors

a.) To establish the character of vector e_α, let the input power which is a scalar quantity, be $P = e_m i^m$. The value of the power input is an invariant (or tensor) that is from physical considerations its value is independent of the coordinate system in which it is measured just like those of the other scalar quantities of output, stored energy etc.

Hence in new coordinates $P = e_\pi i^\pi$. But $i^\pi = C^\pi_m i^m$ from equ. 29. Substituting $P = e_\pi i^\pi = e_\pi C^\pi_m i^m = e_m i^m$. Hence $e_m = e_\pi C^\pi_m$ or

$$e_\pi = C^m_\pi e_m \quad \text{.......................} 37$$

Comparing this equation with equ. 29 it is found that C^m_π which transforms e_m into the new coordinate system is the *inverse* of C^π_m which transforms i^m into the new coordinate system. Hence to differentiate e_α and similar vectors from i^α its index is a lower index. *The vectors that transform by equ. 29 have an upper index and are called "contravariant" vectors, those that are transformed according to equ. 37 have a lower index and are called "covariant" vectors.* Tensors of higher rank also may have any number of upper and lower indices, but for instance $L_{\alpha\beta}$ and $L^{\alpha\beta}$ are two different tensors with different n^2 coefficients.

b.) Since the kinetic energy $(\frac{1}{2})L_{\alpha\beta}i^\alpha i^\beta$ is a scalar invariant, its derivative with respect to i^α that is $L_{\alpha\beta}i^\beta$ is a covariant vector φ_α. Its derivative with respect to i^β, that is $L_{\alpha\beta}$ is a double covariant tensor. Similarly the derivative of the dissipation function, a scalar, $(\frac{1}{2})R_{\alpha\beta}i^\alpha i^\beta$ is $R_{\alpha\beta}i^\beta$, a tensor and also its derivative $R_{\alpha\beta}$ is a double covariant tensor.

Hence the values of L_{mn} and R_{mn} in the new coordinate system with stationary axes are by the formula

$$\boxed{L_{\pi\sigma} = L_{mn}C^m_\pi C^n_\sigma} \quad \text{....................} 38$$

$$L_{\pi\sigma} = \begin{array}{c|ccccc} & d_s & d_r & q_r & q_s & t \\ \hline d_s & L_{sd} & M_d & 0 & 0 & 0 \\ d_r & M_d & L_{rd} & 0 & 0 & 0 \\ q_r & 0 & 0 & L_{rq} & M_q & 0 \\ q_s & 0 & 0 & M_q & L_{sq} & 0 \\ t & 0 & 0 & 0 & 0 & L_{tt} \end{array} \quad \text{........} 39$$

$$R_{\pi\sigma} = \begin{array}{c|c|c|c|c|c|} & d_s & d_r & q_r & q_s & t \\ \hline d_s & r_{sd} & 0 & 0 & 0 & 0 \\ \hline d_r & 0 & r_r & 0 & 0 & 0 \\ \hline q_r & 0 & 0 & r_r & 0 & 0 \\ \hline q_s & 0 & 0 & 0 & r_{sq} & 0 \\ \hline t & 0 & 0 & 0 & 0 & r \\ \hline \end{array} \qquad \ldots\ldots\ldots\ldots 40$$

c.) Among the space vectors so far all but two were found to be tensors. It will be shown now that the remaining two vectors, $L_{\alpha\beta}di^\beta/dt$ which includes *the induced voltage and* $\Gamma_{\alpha\beta,\,\gamma}i^\alpha i^\beta$ which includes *the generated voltage are not tensors,* neither are di^α, di^α/dt or $\Gamma_{\alpha\beta,\,\gamma}$ and their law of transformation does not follow the simple formula of equ. 36. In other words *the space vector, say* di^α *has different lengths and different positions when measured from different sets of coordinate axes, although the vector* i^α *is an invariant.*

XIV. The Transformation Formula of di^α

a.) From equ. 29 $i^m = C^m_\pi i^\pi$. Substituting this value of i^m into di^m

$$di^m = d(i^\pi C^m_\pi) = di^\pi C^m_\pi + i^\pi \frac{\partial C^m_\pi}{\partial x^\sigma}\, dx^\sigma \ldots\ldots\ldots\ldots 41$$

In rotating machinery C^m_π is a function only of $x^t = \theta$. Hence

$$\boxed{di^m = di^\pi C^m_\pi + i^\pi \frac{\partial C^m_\pi}{\partial x^t}\, dx^t} \ldots\ldots\ldots\ldots\ldots\ldots 42$$

The formula shows that *the new vector consists of the old vector* $di^\pi C^m_\pi$ *plus an additional vector* which is a function of the instantaneous velocity etc. If both old and new coordinate axes are stationary the additional vector disappears and di^m becomes a tensor.

b.) A *physical representation* is given in fig. 19. Let the current vector in the rotor change from OA to OB while the axis of the sliprings changes from θ to $\theta + d\theta$.

The observers on the *stationary* axes measure the scalar value of the current before the change along d_r as OC and after the change as OD. Along axis q they measure the change in current as EF. Hence they construct the di^π vector as AB.

The observers on the *moving* axes measure the scalar value of the current before the change along axis a as OG and after the change as

$OI = OL$ instead of OQ, since the axis itself from which the measurement is made has also moved and the projection of OB on the changed axis is OI. It is true that the observers also measure a change of displacement $d\theta$, but that does *not* appear to them as a change of current as measured by an ammeter (it will be shown below that that appears to them as an additional voltage). Similarly along axis b the change of current is $OH - OI = HK$. Hence the resultant di^m as measured from the moving axes is constructed as AM.

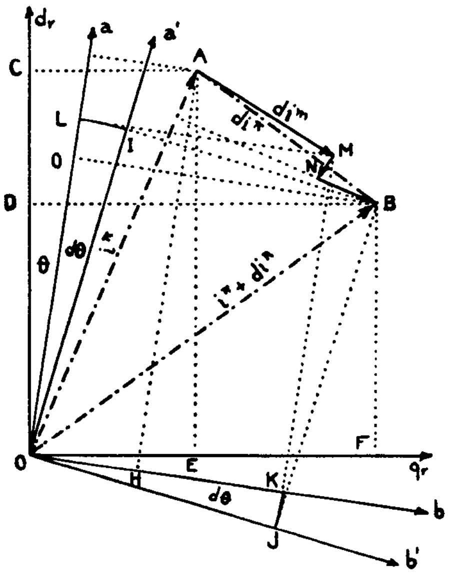


FIG. 19

From equation 42:

1.) $di^m = AM$ is the vector measured from the moving axes

2.) $di^\pi C^m_\pi = AB$ is the vector measured from the stationary axes

3.) $i^\pi dx^\epsilon(\partial C^m_\pi/\partial x^\epsilon) = BN$ is a vector whose magnitude is equal to $i^\pi dx^\epsilon = OA\,.\,LI/OL$ and which is rotated at right angles to i^π by $\partial C^m_\pi/dx^\epsilon$.

The vector NM remaining on the diagram, perpendicular to AB is an infinitesimal of the second order, (its magnitude is $di\,.\,d\theta = AB\,.\,LI/OL$) and would disappear from the drawing if the angle $d\theta$ and the length AB could be assumed small enough.

c.) The transformation formula for di^m/dt is

$$\frac{di^m}{dt} = \frac{di^\pi}{dt}\,C^m_\pi + i^\pi\,\frac{\partial C^m_\pi}{\partial x^\epsilon}\,i^\epsilon \quad\ldots\ldots\ldots\ldots\ldots\ldots 43$$

Also

$$L_{mk}\frac{di^m}{dt} = L_{\pi\sigma}C^\pi_k C^\sigma_m \frac{di^\epsilon}{dt} C^m_\epsilon + L_{\pi\sigma}C^\pi_k C^\sigma_m i^\tau i^\epsilon \frac{\partial C^m_\epsilon}{\partial x^\tau}$$

$$= L_{\pi\epsilon}\frac{di^\epsilon}{dt} C^\pi_k + L_{\pi\sigma}C^\pi_k C^\sigma_m i^\tau i^\epsilon \frac{\partial C^m_\epsilon}{\partial x^\tau} \quad\ldots\ldots\ldots\ldots\ldots 44$$

since $C^\sigma_m C^m_\epsilon = \delta^\sigma_\epsilon =$ unit matrix.

XV. The Transformation Formula of $\Gamma_{\epsilon\sigma,\,\pi}$

In the Equation of Motion (equ. 9)

$$e_k = R_{mk}i^m + L_{mk}\frac{di^m}{dt} + \Gamma_{mn,\,k}i^m i^n$$

e_k is a tensor of rank one, similarly $R_{mk}i^m$. Hence it follows that the *sum* of the remaining two vectors also must be a tensor. Since $L_{mk}di^m/dt$ is not a tensor, therefore $\Gamma_{mn,\,k}i^m i^n$ is not a tensor, neither is $\Gamma_{mn,\,k}$.

The formula for the transformation of $\Gamma_{mn,\,k}$ can be found from the fact that the sum of the above mentioned two vectors is a covariant tensor of rank one. Hence according to equ. 37

$$L_{\pi\epsilon}\frac{di^\epsilon}{dt} + \Gamma_{\epsilon\sigma,\,\pi}i^\epsilon i^\sigma = \left(L_{mk}\frac{di^m}{dt} + \Gamma_{mn,\,k}i^m i^n\right) C^k_\pi \quad\ldots\ldots\ldots 45$$

Substituting the value of $L_{mk}di^m/dt$ from equ. 44

$$L_{\pi\epsilon}\frac{di^\epsilon}{dt} + \Gamma_{\epsilon\sigma,\,\pi}i^\epsilon i^\sigma = L_{\pi\epsilon}\frac{di^\epsilon}{dt} C^\pi_k C^k_\pi + L_{\pi\sigma}\, C^\pi_k C^\sigma_m C^k_\pi i^\tau i^\epsilon \frac{\partial C^m_\epsilon}{\partial x^\tau} + \Gamma_{mn,\,k}i^m i^n\, C^k_\pi$$

$$\Gamma_{\epsilon\sigma,\,\pi}i^\epsilon i^\sigma = \Gamma_{mn,\,k}i^\epsilon\, C^m_\epsilon i^\sigma C^n_\sigma C^k_\pi + L_{mk}\frac{\partial C^m_\epsilon}{\partial x^\sigma}\, C^k_\pi i^\sigma i^\epsilon$$

$$\boxed{\Gamma_{\epsilon\sigma,\,\pi} = \Gamma_{mn,\,k}\, C^m_\epsilon C^n_\sigma C^k_\pi + L_{mk}\frac{\partial C^m_\epsilon}{\partial x^\sigma}\, C^k_\pi} \quad\ldots\ldots\ldots\ldots 46$$

This formula like every other transformation formula is true for any two coordinate systems and is one of the fundamental formulae of the Absolute Calculus. The first part leaves the operator unchanged, while the second part again is a function of the velocity and it disappears if both sets of coordinate axes are stationary.

XVI. The Calculation of $\Gamma_{\epsilon\sigma,\,\pi}$

a.) The calculation of the generalized Christoffel symbol for the representative machine with stationary coordinate axes is made again in two steps by dividing $\Gamma_{\epsilon\sigma,\,\pi}$ into $\Gamma_{\epsilon t,\,\pi}$ and $\Gamma_{\epsilon\sigma,\,t}$.

1.) The value of $\Gamma_{\epsilon t, \pi}$ is found from

$$\Gamma_{\epsilon t,\pi} = \Gamma_{mt,k}\, C^m_\epsilon C^k_\pi + L_{mk} \frac{\partial C^m_\epsilon}{\partial x^t} C^k_\pi \qquad \ldots\ldots 47$$

(C^t_t is unity). As an example $\Gamma_{\epsilon t, \pi}$ will be calculated in detail

$$\frac{\partial C^m_\epsilon}{\partial x^t} = \begin{array}{c|c|c|c|c|c|} \epsilon \backslash m & d_s & a & b & q_s & t \\ \hline d_s & 0 & 0 & 0 & 0 & 0 \\ d_r & 0 & -\sin\theta & -\cos\theta & 0 & 0 \\ q_r & 0 & \cos\theta & -\sin\theta & 0 & 0 \\ q_s & 0 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad \ldots\ldots 48$$

$$L_{mk} \frac{\partial C^m_\epsilon}{\partial x^t} C^k_\pi = \begin{array}{c|c|c|c|c|c|} \epsilon \backslash \pi & d_s & d_r & q_r & q_s & t \\ \hline d_s & 0 & 0 & 0 & 0 & 0 \\ d_r & 0 & 0 & -L_{rq} & -M_q & 0 \\ q_r & M_d & L_{rd} & 0 & 0 & 0 \\ q_s & 0 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \qquad \ldots\ldots 49$$

$$\Gamma_{mt,k}\, C^m_\epsilon C^k_\pi = \begin{array}{c|c|c|c|c|} \epsilon \backslash \pi & d_s & d_r & q_r & q_s \\ \hline d_s & 0 & 0 & -M_d & 0 \\ d_r & 0 & 0 & L_{rq} - L_{rd} & M_q \\ q_r & -M_d & L_{rq} - L_{rd} & 0 & 0 \\ q_s & 0 & M_q & 0 & 0 \\ \hline \end{array} \qquad \ldots\ldots 50$$

$$\Gamma_{\epsilon t,\pi} = \begin{array}{c|c|c|c|c|} \epsilon \backslash \pi & d_s & d_r & q_r & q_s \\ \hline d_s & 0 & 0 & -M_d & 0 \\ d_r & 0 & 0 & -L_{rd} & 0 \\ q_r & 0 & L_{rq} & 0 & 0 \\ q_s & 0 & M_q & 0 & 0 \\ \hline \end{array} \qquad \ldots\ldots 51$$

b.) If $\Gamma_{\epsilon t, \pi}$ is assumed as an operator in the *old* coordinate system, the same value of $\Gamma_{mt, k}$ given in equ. 15 is found by the use of equ. 47 that is by

$$\Gamma_{mt,k} = \Gamma_{\epsilon t,\pi}\, C^\epsilon_m C^\pi_k + L_{\epsilon\pi} \frac{\partial C^\epsilon_m}{\partial x^t} C^\pi_k \qquad \ldots\ldots 52$$

The recalculation of $\Gamma_{mt, k}$ serves as a check on the correctness of $\Gamma_{\epsilon t, \pi}$. That is

$$\Gamma_{\epsilon t,\pi} C^{\epsilon}_{m} C^{\pi}_{k} = \begin{array}{c|c|c|c|c|} & d_s & a & b & q_s \\ \hline d_s & 0 & -M_d \sin\theta & -M_d \cos\theta & 0 \\ \hline a & 0 & (L_{rq} - L_{rd}) \sin\theta \cos\theta & -(L_{rd} \cos^2\theta - L_{rq} \sin^2\theta) & 0 \\ \hline b & 0 & L_{rq} \cos^2\theta + L_{rd} \sin^2\theta & (L_{rd} - L_{rq}) \sin\theta \cos\theta & 0 \\ \hline q_s & 0 & M_q \cos\theta & -M_q \sin\theta & 0 \\ \hline \end{array} \quad ..53$$

$$L_{\epsilon\pi} \frac{\partial C^{\epsilon}_{m}}{\partial x^{t}} C^{\pi}_{k} = \begin{array}{c|c|c|c|c|} & d_s & a & b & q_s \\ \hline d_s & 0 & 0 & 0 & 0 \\ \hline a & -M_d \sin\theta & (L_{rq} - L_{rd}) \sin\theta \cos\theta & L_{rd} \sin^2\theta + L_{rq} \cos^2\theta & M_q \cos\theta \\ \hline b & -M_d \cos\theta & -(L_{rd} \cos^2\theta + L_{rq} \sin^2\theta) & (L_{rd} - L_{rq}) \sin\theta \cos\theta & -M_q \sin\theta \\ \hline q_s & 0 & 0 & 0 & 0 \\ \hline \end{array} \quad ..54$$

the sum of these two matrices gives the matrix of equ. 15.

c.) The value of $\Gamma_{\epsilon\sigma,\,t}$ is found from

$$\Gamma_{\epsilon\sigma,\,t} = \Gamma_{mn,\,t}\, C^m_\epsilon\, C^n_\sigma \qquad \text{..................55}$$

Since $k = t$, m in L_{mk} also must be t and $\partial C^t_t/\partial x^t$ is zero.

$$\Gamma_{\epsilon\sigma,\,t} = \begin{array}{c|c|c|c|c|} & d_s & d_r & q_r & q_s \\ \hline {}_\epsilon d_s & 0 & 0 & M_d/2 & 0 \\ d_r & 0 & 0 & (L_{rd} - L_{rq})/2 & -M_q/2 \\ q_r & M_d/2 & (L_{rd} - L_{rq})/2 & 0 & 0 \\ q_s & 0 & -M_q/2 & 0 & 0 \\ \hline \end{array} \qquad ..56$$

In section XXVII $\Gamma_{\epsilon\sigma,\,t}$ is not given by this matrix but by another identical with that of $\Gamma_{\epsilon t,\,\pi}$ in equ. 51 with opposite signs. The discrepancy however is only apparent. This matrix is used only in the calculations of torque, $\Gamma_{\epsilon\sigma,\,t}i^\epsilon i^\sigma$. Forgetting for the time being about the index t, the expression for torque is a homogeneous quadratic form. That means that only the *symmetrical* part of the matrix is effective, that is the matrix given in equ. 56.

d.) If the labor-saving device is not followed and $\Gamma'_{mn,\,k}$ is defined as in equ. 8 then the coefficients of $\Gamma'_{\epsilon\sigma,\,\pi}$ would be given in three matrices, two for the determination of voltages (instead of one) and one for the torque, as previously. The three components of $\Gamma'_{mn,\,k}$ are given in equs. 20–22.

1.) $\Gamma'_{\epsilon t,\,\pi}$ is found from $\Gamma'_{mt,\,k}$ equ. 47. Hence $\Gamma'_{mt,\,k}C^m_\epsilon C^k_\pi$ is one half of equ. 50 the other term is given in equ. 49. Their sum is

$$\Gamma'_{\epsilon t,\,\pi} = \begin{array}{c|c|c|c|c|} & d_s & d_r & q_r & q_s \\ \hline {}_\epsilon d_s & 0 & 0 & -M_d/2 & 0 \\ d_r & 0 & 0 & -(L_{rd} + L_{rq})/2 & -M_q/2 \\ q_r & M_d/2 & (L_{rd} + L_{rq})/2 & 0 & 0 \\ q_s & 0 & M_q/2 & 0 & 0 \\ \hline \end{array} \qquad ..57$$

2.) $\Gamma'_{t\sigma,\,\pi}$ is found from $\Gamma'_{tn,\,k}$ by equ. 46. The matrix of $\Gamma'_{tn,\,k}C^n_\sigma C^k_\pi$ is one half of equ. 50 the other term is however zero since σ is not equal to t. Hence

$$\Gamma'_{t\sigma,\,\pi} = \begin{array}{c|c|c|c|c|} & d_s & d_r & q_r & q_s \\ \hline {}_\sigma d_s & 0 & 0 & -M_d/2 & 0 \\ d_r & 0 & 0 & (L_{rq} - L_{rq})/2 & M_q/2 \\ q_r & -M_d/2 & (L_{rq} - L_{rd})/2 & 0 & 0 \\ q_s & 0 & M_q/2 & 0 & 0 \\ \hline \end{array} \qquad ...58$$

b.) Since the applied steady-state voltages along the direct and quadrature axes of the armature are constants $e^{dr} = e \sin \delta$ and $e^{qr} = e \cos \delta$, *the steady-state impedance tensor of the group* is found by making $p = 0$. Hence (since $p\theta = v\omega = \omega$ and $\omega L = X$)

$$Z_{\alpha\beta} = \begin{array}{c|c|c|c|c|c|c|}
 & d'_s & d_s & d_r & q_r & q_s & q'_s \\
\hline
d'_s & r'_{sd} & 0 & -X'_{md} \sin \delta & X'_{md} \cos \delta & 0 & 0 \\
d_s & 0 & r_{sd} & 0 & -X_{md} & 0 & 0 \\
d_r & 0 & 0 & \begin{array}{c} r_r + r'_r - \sin \delta . \\ \cos \delta \, (X'_{rq} - X'_{rd}) \end{array} & \begin{array}{c} -X_{rd} - X'_{rd} \cos^2 \delta \\ -X'_{rq} \sin^2 \delta \end{array} & 0 & 0 \\
q_r & 0 & 0 & \begin{array}{c} X_{rq} + X'_{rd} \sin^2 \delta + \\ X'_{rq} \cos^2 \delta \end{array} & \begin{array}{c} r_r + r'_r + \sin \delta . \\ \cos \delta \, (X'_{rq} - X'_{rd}) \end{array} & 0 & 0 \\
q_s & 0 & 0 & X_{mq} & 0 & r_{sq} & 0 \\
q'_s & 0 & 0 & -X'_{mq} \cos \delta & -X'_{mq} \sin \delta & 0 & r'_{sq} \\
\hline
\end{array}$$

If the resistances are assumed to be zero the equations check with those given by Doherty and Nickle.

XXI. Scattered Examples

1.) **Fynn-Weichsel Motor.** Fig. 14. Its transient impedance tensor is

The sum of the two matrices 57 and 58 is the matrix of $\Gamma_{\epsilon t,\,\pi}$ as given in equ. 51. Hence both definitions give the same voltages. That is $\Gamma_{(\epsilon t),\,\pi}i^\epsilon = \Gamma_{(t\epsilon),\,\pi}i^\epsilon = \Gamma_{\epsilon t,\,\pi}i^\epsilon$.

3.) $\Gamma'_{\epsilon\sigma,\,t}$ is identical with $\Gamma_{\epsilon\sigma,\,t}$ given in equ. 56.

It should be noted that $\Gamma'_{\epsilon t,\,\pi}$ is not equal to $\Gamma_{t\sigma,\,\pi}$ or in other words $\Gamma_{\epsilon\sigma,\,\pi}$ *is asymmetrical in the indices* ϵ *and* σ *in either definition of* $\Gamma_{mn,\,k}$.

e.) With the definition of section VIe matrix 51 would have been denoted as $\Gamma_{t\epsilon,\,\pi}$ instead of $\Gamma_{\epsilon t,\,\pi}$.

XVII. The Coriolis Voltage

a.) An interesting physical interpretation can be given to the transformation formula of $\Gamma_{\alpha\beta,\,\gamma}$ in equs. 46 or 47 and 55.

Let the Equation of Voltage (equ. 17) be set up for the representative machine with *stationary* coordinate axes

$$e_\pi = R_{\pi\epsilon}i^\epsilon + L_{\pi\epsilon}\frac{di^\epsilon}{dt} + \Gamma_{\epsilon t,\,\pi}i^\epsilon i^t \dots\dots\dots\dots\dots 59$$

b.) *The observers on the stationary axes measure two voltages* due to the presence of flux lines:

1.) An *induced* voltage $L_{\pi\epsilon}di^\epsilon/dt = d\varphi_\pi/dt$ in all stator and rotor axes assuming:

a.) the rotor conductors stationary,

b.) the currents varying.

2.) A *generated* voltage $\Gamma_{\epsilon t,\,\pi}i^\epsilon i^t = \Psi_{t\pi}i^t$ in all rotor axes assuming:

a.) the currents unvarying,

b.) the rotor conductors cutting the resultant rotor flux-density waves with a velocity i^t.

c.) In changing over to any machine in which one rotor layer has *moving* coordinate axes, not the whole length of the previous vector $L_{\pi\epsilon}di^\epsilon/dt$ appears as an induced voltage, only a part of it since the moving observers measure a smaller di^m (section XIV). The decrease of the induced voltage vector appears however as an additional generated voltage vector. That is *the moving observers measure three voltages:*

1.) An *induced* voltage in all stator and rotor axes, assuming:

a.) the rotor conductors stationary,

b.) the coordinate axes stationary,

c.) the currents varying.

This voltage is smaller than the corresponding voltage measured by the stationary observers.

2.) A *generated* voltage in all rotor axes, assuming:
a.) the currents unvarying,
b.) the coordinate axes stationary,
c.) the rotor conductors moving.
It is equal to the corresponding voltage measured by the stationary observers.

3.) A *generated* voltage in all stator and rotor axes, assuming:
a.) the currents unvarying,
b.) the rotor conductors stationary,
c.) the coordinate axes moving.
Since the currents flowing into the moving coordinate axes are constants, the flux-density wave produced by them also rotates and cuts all stator and rotor windings. Above the moving conductors are cutting the stationary flux-lines, here the moving flux-lines cut the stationary conductors. This latter voltage is not measured by the stationary observers. *This difference in the measured generated voltage is equal to the difference in the measured induced voltage.*

It should also be noted that moving coordinate axes are introduced not only by sliprings on a rotor winding, but also by revolving brushes on a stator winding.

In analogy to its more or less dynamical equivalent, the "Coriolis force" this apparently additional generated voltage may be called the "Coriolis voltage." Also the flux density due to the currents in the moving coordinate axes may be called the "Coriolis flux density."

d.) From the Equation of Voltage (equ. 17) of the machine with moving coordinate axes the induced voltage is $L_{mk}di^m/dt$ the generated voltage is $\Gamma_{mt,\,k}i^m i^t$. Each of these can be divided by their transformation formulae into two components.

1.) By equ. 44 the induced voltage is

$$L_{mk}\frac{di^m}{dt} = L_{\pi\epsilon}\frac{di^\epsilon}{dt}C^\pi_k + L_{\pi\sigma}C^\pi_k C^\sigma_m\frac{\partial C^m_\epsilon}{\partial x^t}i^\epsilon i^t \dots\dots\dots\dots 60$$

The first term on the right hand side is the induced voltage measured by the stationary observers, but expressed along the moving coordinates by C^π_k. The second term is the decrease of the induced voltage.

2.) By equ. 52 the total generated voltage is

$$\Gamma_{mt,\,k}i^m i^t = \Gamma_{\epsilon t,\,\pi}C^\epsilon_m C^\pi_k i^m i^t + L_{\epsilon\pi}\frac{\partial C^\epsilon_m}{\partial x^t}C^\pi_k i^m i^t \dots\dots\dots\dots 61$$

The first term on the right hand side is the generated voltage vector measured by stationary observers but expressed along the moving coordinates by C^{π}_{k}. The second term is the Coriolis voltage, the increase in the generated voltage.

e.) The decrease in the induced voltage is equal to the increase in the generated voltage, that is

$$L_{\pi\sigma} C^{\pi}_{k} C^{\sigma}_{m} \frac{\partial C^{m}_{\epsilon}}{\partial x^{t}} i^{\epsilon} i^{t} + L_{\sigma\pi} C^{\pi}_{k} \frac{\partial C^{\sigma}_{m}}{\partial x^{t}} i^{m} i^{t}$$

$$= L_{\pi\sigma} C^{\pi}_{k} i^{\epsilon} i^{t} \left(\frac{\partial C^{m}_{\epsilon}}{\partial x^{t}} C^{\sigma}_{m} + \frac{\partial C^{\sigma}_{m}}{\partial x^{t}} C^{m}_{\epsilon} \right) = L_{\pi\sigma} C^{\pi}_{k} i^{\epsilon} i^{t} \frac{\partial (C^{m}_{\epsilon} C^{\sigma}_{m})}{\partial x^{t}} = 0$$

since $C^{m}_{\epsilon} C^{\sigma}_{m}$ is the idemfactor and its derivative is zero. Hence from equ. 56

$$\text{Coriolis voltage} = \bar{e}_{k} = L_{\epsilon\pi} C^{\pi}_{k} \frac{\partial C^{\epsilon}_{m}}{\partial x^{t}} i^{m} i^{t} = \Phi_{kt} i^{t} \quad \ldots\ldots\ldots 62$$

f.) The sum of equ. 60 and equ. 61 gives

$$L_{mk} \frac{di^{m}}{dt} + \Gamma_{mt,k} i^{m} i^{t} = \left(L_{\pi\epsilon} \frac{di^{\epsilon}}{dt} + \Gamma_{\epsilon t,\pi} i^{\epsilon} i^{t} \right) C^{\pi}_{k} \quad \ldots\ldots\ldots 63$$

which is the reverse transformation of equ. 45.

CONSTANT SPEED

XVIII. The Transient Impedance Matrix

a.) In by far the largest number of problems the speed of the rotor is assumed to be known. In that case the Equation of Voltage is simplified to

$$e_{\alpha} = R_{\alpha\beta} i^{\beta} + L_{\alpha\beta} \frac{di^{\beta}}{dt} + \Gamma_{\beta t,\alpha} i^{\beta} i^{t}$$

$$= (R_{\alpha\beta} + L_{\alpha\beta} p + \Gamma_{\beta t,\alpha} i^{t})\, i^{\beta}$$

where $p = d/dt$. The expression in parenthesis is an important dyadic, is denoted by $Z_{\alpha\beta}$ and called the "transient impedance matrix" representing the opposition of a machine to suddenly applied terminal voltages while its speed is maintained constant, that is

$$\boxed{Z_{\alpha\beta} = R_{\alpha\beta} + L_{\alpha\beta} p + \Gamma_{\beta t,\alpha} i^{t}} \quad \ldots\ldots\ldots\ldots\ldots 64$$

Hence for steady speed the Equation of Voltage is

$$\boxed{e_\alpha = Z_{\alpha\beta} i^\beta} \qquad \ldots\ldots 65$$

from which the current is found by calculating the inverse of $Z_{\alpha\beta} = Y^{\alpha\beta}$

$$\boxed{i^\beta = e_\alpha Y^{\alpha\beta}} \qquad \ldots\ldots 66$$

$Y^{\alpha\beta}$ may be called the "admittance matrix." (See section XXXId.)

b.) For the representative machine with stationary coordinate axes the impedance matrix is

$$Z_{\alpha\beta} = \begin{array}{c|c|c|c|c|} & d_s & d_r & q_r & q_s \\ \hline d_s & r_{sd} + L_{sd}p & M_d p & -M_d p\theta & 0 \\ d_r & M_d p & r_r + L_{rd}p & -L_{rd}p\theta & 0 \\ q_r & 0 & L_{rq}p\theta & r_r + L_{rq}p & M_q p \\ q_s & 0 & M_q p\theta & M_q p & r_{sq} + L_{sq}p \\ \hline \end{array} \qquad \ldots 67$$

It should be noted that $Z_{\alpha\beta}$ is not a symmetrical matrix. For the representative machine with two layers of stator and rotor windings to be used in general work, the transient impedance matrix is given in Table I.

c.) *For any machine whose connection tensor is given by C^π_α its transient impedance matrix is found from that of the above representative machine by*

$$\boxed{Z_{\alpha\beta} = Z_{\pi\sigma} C^\pi_\alpha C^\sigma_\beta + L_{\pi\sigma} \frac{\partial C^\pi_\beta}{\partial x^t} C^\sigma_\alpha i^t} \qquad \ldots\ldots 68$$

as follows from equs. 64 and 46. If no slip-rings exist the last term disappears and $Z_{\alpha\beta}$ is a tensor. The torque is by equ. 19

$$\boxed{\text{Torque} = \Gamma_{\alpha\beta,t} i^\alpha i^\beta} \qquad \ldots\ldots 69$$

where $\Gamma_{\alpha\beta,\,t}$ is found from $\Gamma_{\pi\sigma,\,t}$ given in Table I by the transformation formula (equ. 55)

$$\Gamma_{\alpha\beta,t} = \Gamma_{\pi\sigma,t} C^\pi_\alpha C^\sigma_\beta \qquad \ldots\ldots 70$$

When the impressed voltage is *a–c* then $p = d/dt$ is replaced by $j\omega$, pL becomes jX and the impedance matrix may be called the "steady-state impedance matrix." With an applied *d–c e.m.f.* $p = 0$. Also $p\theta$ becomes $v\omega$ where v is the ratio of the actual speed to the synchronous speed.

d.) *When any number and any types of rotating machines are connected in any manner whatever the transient impedance of the group is equal to the sum of the transient impedances of the individual units,* transformed by the connection tensor of the group C^{α}_{γ}. That is

$$\boxed{Z_{\gamma\delta} = (Z'_{\alpha\beta} + Z''_{\alpha\beta} + Z'''_{\alpha\beta} + \ldots.)\, C^{\alpha}_{\gamma} C^{\beta}_{\delta}} \qquad \ldots\ldots\ldots 71$$

e.) Even when only the steady-state performance is desired it is always a safe procedure to set up first the transient impedance matrix. *The transient impedance matrix $Z_{\alpha\beta}$ gives the instantaneous voltage appearing at each terminal due to a suddenly applied unit current at any one of the terminals (with the speed maintained constant), while the transient admittance matrix $Y^{\alpha\beta}$ gives the instantaneous current flowing through each terminal due to a suddenly applied unit voltage at any one of the terminals.*

XIX. The Repulsion Motor

a.) As an example let the performance of the repulsion motor be calculated whose connection diagram and connection tensor are given in fig. 6. Its *transient impedance tensor* is

$$Z_{\alpha\beta} = \begin{array}{c|c|c|} & d_s & a \\ \hline d_s & r_{sd} + L_{sd}p & M_d\,(\cos \alpha p - \sin \alpha p\theta) \\ \hline a & M_d \cos \alpha p & r_r + (L_{rd}\cos^2\alpha + L_{r_q}\sin^2\alpha)p - (L_{rd} - L_{r_q})\sin\alpha\cos\alpha p\theta \\ \hline \end{array}$$

If the airgap is uniform, $L_{r_q} = L_{rd} = L_r$, the impedance tensor simplifies to

$$Z_{\alpha\beta} = \begin{array}{c|c|c|} & d_s & a \\ \hline d_s & r_{sd} + L_{sd}p & M_d\,(\cos \alpha p - \sin \alpha p\theta) \\ \hline a & M_d \cos \alpha p & r_r + L_r p \\ \hline \end{array}$$

If the determinant is denoted by D, the *transient admittance tensor* is

$$Y^{\alpha\beta} = \begin{array}{c|c|c|} & d_s & a \\ \hline d_s & (r_r + L_r p)/D & -\,M_d\,(\cos \alpha p - \sin \alpha p\theta)/D \\ \hline a & -\,M_d \cos \alpha p/D & (r_{sd} + L_{sd}p)/D \\ \hline \end{array}$$

b.) With a suddenly applied terminal voltage $e_{ds} = e1$ where 1 is the Heaviside unit function, $i^{ds} = e_{ds}\, Z^{ds\, ds}$ and $i^{a} = e_{ds}\, Z^{ds\, a}$ or if $p\theta = v\omega$

$$i^{ds} = \frac{e_{ds}\,(r_r + L_r p)1}{r_r r_{sd} + p(r_r L_{sd} + r_{sd}L_r + M_d^2 \cos\alpha \sin \alpha v\omega) + p^2(L_{sd}L_r - M_d^2 \cos^2 v\omega}$$

c.) With an a–c terminal voltage $e_{ds} = e \sin \omega t = \bar{e}$ the *steady-state admittance tensor* is

$$Z^{\alpha\beta} = \begin{array}{c|c|c|} & d_s & a \\ \hline d_s & (r_r + jX_r)/D & (-jX_m \cos \alpha + X_m \sin \alpha v)/D \\ \hline a & -jX_m \cos \alpha/D & (r_{sd} + jX_{sd})/D \\ \hline \end{array}$$

where

$$D = (r_r + r_{sd} + X_m{}^2 \cos^2 \alpha v - X_{sd}X_r) + j(r_rX_{sd} + r_{sd}X_r + X_m{}^2 \sin \alpha \cos \alpha v)$$

hence

$$i^{ds} = \bar{e}(r_r + jX_r)/D \text{ and } i^a = -\bar{e}jX_m (\cos \alpha + jv \sin \alpha)/D.$$

d.) The value of $\Gamma_{\alpha\beta,\, t}$ is

$$\Gamma_{\alpha\beta,\, t} = \begin{array}{c|c|c|} & d_s & a \\ \hline d_s & 0 & M_d \sin \alpha/2 \\ \hline a & M_d \sin \alpha/2 & (L_{rd} - L_{rq}) \sin \alpha \cos \alpha/2 \\ \hline \end{array}$$

For a machine with smooth airgap $\Gamma_{\alpha\beta,\, t} = \Gamma_{ds\, a,\, t} + \Gamma_{a\, ds,\, t}$. The torque is $2\Gamma_{ds\, a,\, t}\, i^{ds} i^a$. Since the steady-state torque is to be expressed in synchronous watts the expression should be multiplied by ω. Hence

$$\text{Torque} = (X_m \sin \alpha) \cdot [\bar{e}(r_r + jX)/D] \cdot [\bar{e}X_m(v \sin \alpha - j \cos \alpha)/D].$$

Each expression is a time vector due to the presence of j, hence the products represent scalar products. Since the scalar product $(a + jb) \cdot (c + jd) = ac + bd$

$$\text{Torque} = \frac{\bar{e}^2 X_m^2 \sin \alpha\, (r_r \sin \alpha - X_r \cos \alpha)}{(r_r + r_{sd} + X_m^2 \cos^2 \alpha v - X_{sd}X_r)^2 + (r_rX_{sd} + r_{sd}X_r + X_m^2 \sin \alpha \cos \alpha v)^2}$$

XX. Two Salient-pole Synchronous Machines

a.) The connection diagram and connection tensor of the group is given in fig. 17. *The transient impedance tensor of the group* is found from equ. 71. The constants of the second machine are primed. Also $\delta_{II} - \delta_I = \delta$. Hence $Z_{\alpha\beta}$ is (since $i^b = -i^{b'}$).

	d'_s	d_s	d_r	q_r	q_s	q'_s
d'_s	$r'_{sd} + L'_{sd}p$	0	$-M'_d(\cos \delta p + \sin \delta p\theta')$	$-M'_d\ (\sin \delta p - \cos \delta p\theta')$	0	0
d_s	0	$r_{sd} + L_{sd}p$	$M_d p$	$-M_d\, p\theta$	0	0
d_r	$-M'_d \cos \delta p$	$M_d p$	$r_r + r'_r + (L_{rd} + L'_{rd} \cos^2 \delta + L'_{rq} \sin^2 \delta)\, p + (L'_{rd} - L'_{rq}) \sin \delta \cos \delta p\theta'$	$(L'_{rd} - L'_{rq}) \sin \delta \cos \delta p - (L'_{rd} \cos^2 \delta + L'_{rq} \sin^2 \delta)\, p\theta' - L_{rd}\, p\theta$	0	$M'_q \sin \delta p$
q_r	$-M'_d \sin \delta p$	0	$(L'_{rd} - L'_{rq}) \sin \delta \cos \delta p + (L'_{rd} \sin^2 \delta + L'_{rq} \cos^2 \delta)\, p\theta' + L_{rq}\, p\theta$	$r_r + r'_r + (L_{rq} + L'_{rd} \sin^2 \delta + L'_{rq} \cos^2 \delta)\, p - (L'_{rd} - L'_{rq}) \sin \delta \cos \delta p\theta'$	$M_q\, p$	$-M'_q \cos \delta p$
q_s	0	0	$M_q p\theta$	$M_q\, p$	0	0
q'_s	0	0	$-M'_q\ (\cos \delta p\theta' + \sin \delta p)$	$-M'_q\ (\sin \delta p\theta' - \cos \delta p)$	0	$r'_{sq} + L'_{sq}\, p$

	f	d_r^2	q_r^2	q_s
f	$(r'_r + L'_r\,p) + (r_s + L_s\,p)\,n^2 + 2Mn\cos\alpha p - Mn\sin\alpha p\theta$	$L'_r\cos\alpha p + L'_r\sin\alpha p\theta + Mnp$	$L'_r\sin\alpha p - L'_r\cos\alpha p\theta - Mnp$	$M\sin\alpha p$
d_r^2	$Mnp + L'_r\,(\cos\alpha p - \sin\alpha p\theta)$	$r_r^2 + L_r^2 p$	$-L_r^2 p\theta$	0
q_r^2	$L'_r\,(\sin\alpha p + \cos\alpha p\theta)$	$L_r^2 p\theta$	$r_r^2 + L_r^2 p$	Mnp
q_s	$Mn\,(\sin\alpha p + \cos\alpha p\theta)$	$Mnp\theta$	Mnp	$(r_s + L_s p)\,n^2$

For the calculation of torque

$$\Gamma_{\alpha\beta,\,t} = \begin{array}{c|c|c|c|} & f & d_r^2 & q_r^2 \\ \hline f & -Mn\sin\alpha & 0 & -Mn \\ \hline q_s' & Mn\cos\alpha & Mn & 0 \\ \hline \end{array}$$

a.) For its steady-state performance as an *induction motor* $p = j\omega(1 - v)$ and $p\theta = v\omega$.

b.) For its performance as a *synchronous motor* $p = 0$ and $p\theta = \omega$.

2.) **Schrage Motor.** Fig. 8. Its connection tensor as a polyphase motor is

$$C_\alpha^\pi = \begin{array}{c|c|c|c|c|c|c|} & d_s^1 & d_r^1 & d_r^2 & q_r^2 & q_r^1 & q_s \\ \hline f & n & \epsilon^{k\beta} - \epsilon^{k\alpha} & 0 & 0 & -k\,(\epsilon^{k\beta} - \epsilon^{k\alpha}) & -kn \\ \hline a & 0 & 0 & \epsilon^{j\theta} & -j\epsilon^{j\theta} & 0 & 0 \\ \hline \end{array}$$

If $(\beta - \alpha)/2 = \gamma$ and $(\alpha + \beta)/2 = \delta$

$$Z_{\alpha\beta} = \begin{array}{c|c|c|} & f & a \\ \hline f & \begin{array}{c}(r_s + jsX_s)\,n^2 + 2X_m \sin\gamma\,(\epsilon^{-j\delta} - s\epsilon^{j\delta})\,n \\ +\,4\,(r_r^1 + jX_r^1)\sin^2\gamma\end{array} & jX_m n - 2X_r^1 \sin\gamma\epsilon^{j\delta} \\ \hline a & jsX_m n + 2X_r^1 \sin\gamma\epsilon^{-j\delta} & r_r^2 + jX_r^2 \\ \hline \end{array}$$

3.) **Polyphase Induction Motor** with slip-rings. Fig. 3. Its transient impedance tensor is

$$Z_{\alpha\beta} = \begin{array}{c|c|c|} & d_s & a \\ \hline d_s & r_s + L_s p & Mp\,(\cos\theta - j\sin\theta) \\ \hline a & Mp\,(\cos\theta + k\sin\theta) & r_r + L_r p \\ \hline \end{array}$$

4.). **Frequency-Converter.** Fig. 12. With fundamental-frequency voltage on the slip-rings and slip-frequency voltage on the brushes its transient impedance tensor is

$$Z_{\alpha\beta} = \begin{array}{c|c|c|} & a & d_r \\ \hline a & r_r + L_r p & (r_r + L_r p)\,(\cos\theta - j\sin\theta) \\ \hline d_r & (r_r + L_r p)\,(\cos\theta + k\sin\theta) & r_r + L_r p - j\,L_r p\theta \\ \hline \end{array}$$

5.) **Induction Motor and Frequency-Converter** connected to be used as **Phase-Advancer.** Fig. 15. Its transient impedance tensor is found simply by adding the two previous impedance tensors and multiplying their sum with the connection tensor of the group according to equ. 71. The steady-state impedance tensor of the group is

$$Z_{\alpha\beta} = \begin{array}{c|c|c|c|} & d_s & f & a \\ \hline d_s & r_s + jX_s & jsX_m & 0 \\ \hline f & jX_m & (r_r + jsX_r) + (r_r' + jX_r') & (r_r' + jX_r')\,\epsilon^{j\alpha} \\ \hline a & 0 & (r_r' + jX_r')\,\epsilon^{j\alpha} & r_r' + jX_r' \\ \hline \end{array}$$

NON-HOLONOMIC DYNAMICAL SYSTEMS

XXII. The Equation of Motion of Holonomic Systems

a.) Comparing the definition of $\Gamma_{\alpha\beta,\,\gamma}$ as given in equations 12, 13 and 14 for the representative machine with *moving* coordinate axes and its value given in equ. 51 and 56 for the representative machine with *stationary* coordinate axes, it can be seen that $\Gamma_{\epsilon t,\,\pi}$ is not equal to $\partial L_{\epsilon\pi}/\partial x^t$ neither is $\Gamma_{\epsilon\sigma,\,t}$ equal to $-(\frac{1}{2})\partial L_{\epsilon\sigma}/\partial x^t$ since every coefficient of $L_{\pi\sigma}$ is a constant (see equ. 39). That is, the following important fact should be noted:

The generalized Christoffel symbol is defined in terms of the metric tensor $L_{\alpha\beta}$ by equ. 11 only for the representative machine with moving coordinate axes. For every other coordinate system its value is found by a transformation of coordinates from equ. 46 since as yet $\Gamma_{\alpha\beta,\,\gamma}$ has not been defined for them in terms of the metric tensor.

This is the reason why the qualifying term "generalized" is added to differentiate it from the ordinary Christoffel symbol of Riemannian Geometry which is defined by an equation identical with equ. 8 for *every* coordinate system and which of course also obeys the rule of transformation of equ. 46.

This distinction in the definition of $\Gamma_{\alpha\beta,\,\gamma}$ in terms of $L_{\alpha\beta}$ for the two coordinate systems is due to the fact that *among the innumerable types of rotating electrical machines there is one and one only whose coordinates are true Lagrangian coordinates.* This one is the representative machine with moving coordinate axis having any number of layers of windings on the stator and rotor, since in *order that an axis should be a true Lagrangian coordinate axis it must be connected to the moving conductors.* To this type belong only the synchronous machine and the slipring induction motor provided the sliprings are taken as the coordinate axes. It is only for these axes that *Maxwell's form of the Equation of Voltage* (derived from equs. 5 or 17)

$$\boxed{e_\alpha = R_{\alpha\beta}i^\beta + \frac{d\varphi_\alpha}{dt} = R_{\alpha\beta}i^\beta + \frac{d(L_{\alpha\beta}i^\beta)}{dt} = R_{\alpha\beta}i^\beta + L_{\alpha\beta}\frac{di^\beta}{dt} + \frac{dL_{\alpha\beta}}{dt}i^\beta} \quad ..72$$

is valid. This equation is not valid for any other axes since it gives only the induced and Coriolis voltages, but it does not give the generated voltage due to the motion of rotor conductors. Any attempt of *routine substitution* fails, which immediately can be seen in case of machines with stationary coordinate axes where $dL_{\alpha\beta}/dt = i^t\partial L_{\alpha\beta}/\partial x^t = 0$ as mentioned at the beginning of this section.

This fact has not been recognized before and this is the reason why numerous treatises and papers on rotating machinery start with Hamilton's, Lagrange's or Maxwell's equation of the usual form, but use them only for the alternator with moving coordinate axes and *for no other machine.* They all are compelled to analyze the complicated physical phenomena inside of each type of machine and even in the alternator if its moving axes are changed to stationary axes along the fieldpole and the interpolar space, since the *routine substitution, which after all is the whole purpose of all generalized equations*, fails.

b.) In the following sections $\Gamma_{\alpha\beta,\,\gamma}$ will be defined in terms of the metric tensor for all coordinate systems. The three matrices of $\Gamma_{t\beta,\,\gamma}$ $\Gamma_{\alpha t,\,\gamma}$ and $\Gamma_{\alpha\beta,\,t}$ will be found to be defined slightly differently from the matrices defined previously, but again the voltages and torques calculated from them are identical with those found by previous definitions of $\Gamma_{\alpha\beta,\,\gamma}$.

Also in the next three sections all rotating machines will be considered derived from the representative machine with *moving* coordinate axes, that is their connection tensor C^m_α will be the product of the connection tensor C^π_α given in figs. 2–17 with that of the representative machine with stationary coordinate axes C^m_π. In other words C^m_α will be equal to $C^\pi_\alpha C^m_\pi$.

XXIII. The Equation of Motion of Non-holonomic Systems

a.) Let the Equation of Motion for *true* coordinates be used in the form of equ. 6 that is, let

$$\boxed{f_k = R_{mk}\dot{x}^m + a_{mk}\frac{d\dot{x}^m}{dt} + \left(\frac{\partial a_{mk}}{\partial x^n} - \frac{1}{2}\frac{\partial a_{mn}}{\partial x^k}\right)\dot{x}^m\dot{x}^n} \quad \ldots\ldots\ldots 73$$

Instead of defining the term in parenthesis as $\Gamma_{mn,\,k}$ and calculating its transformation formula (as has been done in section XV) let the whole equation be transformed term by term to a new non-holonomic system with axes ϵ, π, σ ... by replacing a_{mk} by $a_{\alpha\sigma}C^\alpha_m C^\sigma_k$, $\dot{x}^m$ by $\dot{x}^\mu C^m_\mu$ etc. where C^α_m is *not* equal to $\partial x^\alpha/\partial x^m$

$$f_\sigma C^\sigma_k = R_{\alpha\sigma} C^\alpha_m C^\sigma_k \dot{x}^\mu C^m_\mu + a_{\alpha\sigma} C^\alpha_m C^\sigma_k \frac{d\dot{x}^\mu}{dt} C^m_\mu + a_{\alpha\sigma} C^\alpha_m C^\sigma_k \dot{x}^\mu \frac{\partial C^m_\mu}{\partial x^n} \dot{x}^n$$

$$+ \frac{\partial a_{\alpha\sigma}}{\partial x^n} C^\alpha_m C^\sigma_k \dot{x}^\mu C^m_\mu \dot{x}^\gamma C^n_\gamma + a_{\alpha\sigma} \frac{\partial C^\alpha_m}{\partial x^n} C^\sigma_k \dot{x}^\mu C^m_\mu \dot{x}^\gamma C^n_\gamma + a_{\alpha\sigma} \frac{\partial C^\sigma_k}{\partial x^n} C^\alpha_m \dot{x}^\mu C^m_\mu \dot{x}^\gamma C^n_\gamma$$

$$- \frac{1}{2} \frac{\partial a_{\alpha\delta}}{\partial x^k} C^\alpha_m C^\delta_n \dot{x}^\mu C^m_\mu \dot{x}^\gamma C^n_\gamma - \frac{1}{2} a_{\alpha\delta} \frac{\partial C^\alpha_m}{\partial x^k} C^\delta_n \dot{x}^\mu C^m_\mu \dot{x}^\gamma C^n_\gamma$$

$$- \frac{1}{2} a_{\alpha\delta} \frac{\partial C^\delta_n}{\partial x^k} C^\alpha_m \dot{x}^\mu C^m_\mu \dot{x}^\gamma C^n_\gamma \qquad \ldots\ldots 74$$

But

$$a_{\alpha\sigma} C^\sigma_k \dot{x}^\mu \dot{x}^n \left(\frac{\partial C^m_\mu}{\partial x^n} C^\alpha_m + \frac{\partial C^\alpha_m}{\partial x^n} C^m_\mu \right) = a_{\alpha\sigma} C^\sigma_k \dot{x}^\mu \dot{x}^n \frac{\partial\,(C^m_\mu C^\alpha_m)}{\partial x^n} = 0$$

and

$$- \frac{1}{2} a_{\alpha\delta} \dot{x}^\mu C^m_\mu \dot{x}^\gamma C^n_\gamma \left(\frac{\partial C^\alpha_m}{\partial x^k} C^\delta_n + \frac{\partial C^\delta_n}{\partial x^k} C^\alpha_m \right) = - a_{\alpha\delta} \dot{x}^\mu C^m_\mu \dot{x}^\gamma C^n_\gamma \frac{\partial C^\delta_n}{x^k} C^\alpha_m$$

since α and δ also m and n can be interchanged in this expression. Multiplying every term by C^k_σ the result is

$$\boxed{f_\sigma = R_{\alpha\sigma} \dot{x}^\alpha + a_{\alpha\sigma} \frac{d\dot{x}^\alpha}{dt} + \left[\left(\frac{\partial a_{\alpha\sigma}}{\partial x^\gamma} - \frac{1}{2} \frac{\partial a_{\alpha\gamma}}{\partial x^\sigma} \right) + a_{\alpha\delta} C^n_\gamma C^k_\sigma \left(\frac{\partial C^\delta_k}{\partial x^n} - \frac{\partial C^\delta_n}{\partial x^k} \right) \right] \dot{x}^\alpha \dot{x}^\gamma} \quad ..75$$

This is the Equation of Motion for a non-holonomic dynamical system and this is also the form of equation that should be used for the study of various types of rotating machines. The correctness of the formula can be checked by replacing $a_{\alpha\sigma}\dot{x}^\alpha$ by $\partial \bar{T}/\partial \dot{x}^\sigma$ where $\bar{T}$ is the kinetic energy of the system in terms of new differentials, that is $\bar{T} = (\frac{1}{2}) a_{\alpha\sigma} \dot{x}^\alpha \dot{x}^\sigma$. Hence

$$\boxed{\frac{d}{dt}\left(\frac{\partial \bar{T}}{\partial \dot{x}^\sigma} \right) - \frac{\partial \bar{T}}{\partial x^\sigma} + \frac{\partial \bar{T}}{\partial \dot{x}^\delta} \left(\frac{\partial C^\delta_k}{\partial x^n} - \frac{\partial C^\delta_n}{\partial x^k} \right) C^k_\sigma C^n_\gamma \dot{x}^\gamma + \frac{\partial \bar{F}}{\partial \dot{x}^\sigma} = f_\sigma} \quad \ldots .76$$

This is the *modified Equation of Motion of Lagrange* valid for non-holonomic dynamical systems as given by Whittaker in his "Analytical Dynamics," with the notation slightly changed and $\partial \bar{F}/\partial \dot{x}^\sigma$ added.

It should be noted that in equ. 75 all $a_{\alpha\beta}$ and C^k_α are functions of the old coordinates and in the *n dynamical equations there are 2n variables.* They are the n old coordinates and the n new differentials or velocities. The other n equations are $dx^\sigma = C^\sigma_k dx^k$.

b.) If instead of transforming equ. 73 term by term the expression in parenthesis is defined as $\Gamma_{mn,k}$ and transformed just as in section XV the result is

$$f_\pi = R_{\epsilon\pi}\dot{x}^\epsilon + a_{\pi\epsilon}\frac{d\dot{x}_\epsilon}{dt} + \left(\Gamma_{mn,k}\, C^m_\epsilon C^n_\sigma C^k_\pi + a_{mk}\frac{\partial C^m_\epsilon}{\partial x^\sigma} C^k_\pi\right)\dot{x}^\sigma\dot{x}^\epsilon.$$

Even though in the general case of non-holonomic systems the expression in parenthesis is not a function of the n new coordinates but of the n old coordinates and the n new differentials, still *a geometry can be defined, the so called "non-holonomic" geometry in which the expression in the parenthesis plays the rôle of the "coefficients of connection."* Schouten defines

$$\boxed{\Lambda_{\epsilon\sigma,\pi} = \Gamma_{mn,k}\, C^m_\epsilon C^m_\sigma C^k_\pi + a_{mk}\frac{\partial C^m_\epsilon}{\partial x^\sigma} C^k_\pi} \quad \ldots\ldots\ldots\ldots 77$$

so that the *Equation of Motion of non-holonomic dynamical systems* becomes

$$\boxed{a_{\pi\epsilon}\frac{d\dot{x}^\epsilon}{dt} + \Lambda_{\epsilon\sigma,\pi}\dot{x}^\sigma\dot{x}^\epsilon + R_{\epsilon\pi}\dot{x}^\epsilon = f_\pi} \quad \ldots\ldots\ldots\ldots 78$$

XXIV. The Equation of Motion of Quasi-holonomic Systems

a.) *Due to the special character of the coordinates of electrical machinery, equ. 75 and 78 assume special forms.* All $L_{\pi\epsilon}$ and C^k_π are functions of one variable x^t which is both an old and a new coordinate and so *in the n dynamical equations there are only n unknowns, the n new coordinates.* Hence equ. 75 and 78 must be identical with the Equation of Motion as derived in equ. 10 for any coordinate system, that is it must be identical with

$$\boxed{e_\sigma = R_{\alpha\sigma}i^\alpha + L_{\alpha\sigma}\frac{di^\alpha}{dt} + \Gamma_{\gamma\alpha,\sigma}i^\alpha i^\gamma} \quad \ldots\ldots\ldots\ldots 79$$

Hence the generalized Christoffel symbol, must be equal in any coordinate system to

$$\boxed{\Gamma_{\gamma\alpha,\sigma} = \left(\frac{\partial L_{\alpha\sigma}}{\partial x^\gamma} - \frac{1}{2}\frac{\partial L_{\alpha\gamma}}{\partial x^\sigma}\right) + L_{\alpha\delta}C^n_\gamma C^k_\sigma\left(\frac{\partial C^\delta_k}{\partial x^n} - \frac{\partial C^\delta_n}{\partial x^k}\right)} \quad \ldots\ldots 80$$

This is the equation in which $\Gamma_{\alpha\gamma,\sigma}$ is expressed in terms of the metric tensor of the various machines and not equ. 11. This equation is a generalized form of equ. 11 and is valid for any coordinate system. In the special case of true coordinates the coordinates k and n can be interchanged, the last term is zero and equ. 11 is left.

b.) The first expression in parenthesis on the right hand side is the ordinary Christoffel symbol. Its formula of transformation is the same as that of $\Gamma_{\alpha\gamma,\sigma}$ that is equ. 46.

$L_{\alpha\sigma} di^\alpha/dt$ and the Christoffel symbol together form a tensor (see section XV) and since the first three members of the right hand term of equ. 75 are tensors, the last member also must be a tensor. That is, the coefficient of $i^\alpha i^\gamma$ is a tensor of rank three (the "torsion" tensor).

$$\boxed{T_{\gamma\alpha\sigma} = L_{\alpha\delta} C^n_\gamma C^k_\sigma \left(\frac{\partial C^\delta_k}{\partial x^n} - \frac{\partial C^\delta_n}{\partial x^k}\right)} \qquad \ldots\ldots\ldots\ldots\ldots 81$$

It is important to note that $T_{\alpha\gamma\sigma}$ is skew-symmetric in γ and σ that is

$$\boxed{T_{\gamma\alpha\sigma} = - T_{\sigma\alpha\gamma}} \qquad \ldots\ldots\ldots\ldots\ldots 82$$

Hence the generalized Christoffel symbol can be expressed as the sum of the ordinary Christoffel symbol and a tensor of rank three,

$$\boxed{\Gamma_{\gamma\alpha,\sigma} = [\gamma\alpha,\sigma] + T_{\gamma\alpha\sigma}} \qquad \ldots\ldots\ldots\ldots\ldots 83$$

c.) The Equation of Motion of rotating electrical machinery can be written in terms of the new symbols as

$$\boxed{e_\sigma = R_{\alpha\sigma} i^\alpha + L_{\alpha\sigma}\frac{di^\alpha}{dt} + [\gamma\alpha,\sigma] i^\gamma i^\alpha + T_{\gamma\alpha\sigma} i^\gamma i^\alpha} \qquad \ldots\ldots\ldots 84$$

PHYSICAL INTERPRETATIONS

XXV. The Equation of Voltage

a.) The Equation of Voltage can be set up by making σ assume all indices except t. Because in C^k_σ the index k also must assume any value but t the negative terms in equ. 75 disappear and the *Equation of Voltage* is

$$e_\sigma = R_{\alpha\sigma} i^\alpha + L_{\alpha\sigma}\frac{di^\alpha}{dt} + \left(\frac{\partial L_{\alpha\sigma}}{\partial x^\gamma} + L_{\alpha\delta} C^n_\gamma C^k_\sigma \frac{\partial C^\delta_k}{\partial x^n}\right) i^\gamma i^\alpha \ldots\ldots 85$$

b.) The value of $T_{\gamma\alpha\sigma}$ is if $\gamma = t$

$$T_{t\alpha\sigma} = L_{\alpha\delta}\frac{\partial C^\delta_k}{\partial x^t} C^k_\sigma \ldots\ldots\ldots\ldots\ldots 86$$

and if $\alpha = t$ the value of $T_{\gamma\alpha\sigma}$ is

$$T_{\gamma t\sigma} = 0 \qquad \text{.......................... 87}$$

These equations are valid for any machine with stationary or moving coordinate axes since $T_{\alpha\gamma\sigma}$ is a tensor.

c.) *The value of the Christoffel symbol is zero for machines with stationary coordinate axes since all coefficients of $L_{\alpha\beta}$ are constants.* Its value for machines with moving coordinate axes is:

1.) If $\gamma = t$ then from equ. 85

$$[t\alpha, \sigma] = \frac{\partial L_{\alpha\sigma}}{\partial x^t} \qquad \text{.......................... 88}$$

2.) If $\alpha = t$ then from equ. 85

$$[\gamma t, \sigma] = 0 \qquad \text{.......................... 89}$$

Hence in terms of this symbol the *Equation of Voltage* is

$$e_\sigma = R_{\alpha\sigma} i^\alpha + L_{\alpha\sigma} \frac{di^\alpha}{dt} + [t\alpha, \sigma] i^\alpha i^t + T_{t\alpha\sigma} i^\alpha i^t \qquad \text{.......... 90}$$

Comparing it with equ. 10

$$\Gamma_{t\alpha,\sigma} = [t\alpha, \sigma] + T_{t\alpha\sigma} \qquad \text{.................... 91}$$

d.) As an example let the value of $\Gamma_{t\alpha,\sigma}$ be calculated for the representative machine with stationary coordinate axes from equ. 91. Since the coordinate axes are stationary $[t\alpha, \sigma]$ is zero. $T_{t\alpha\sigma}$ is found from equ. 86

1.) $L_{\alpha\delta}$ is given in equ. 39.
2.) C_k^δ is given in equ. 28.
3.)

$$\frac{\partial C_k^\delta}{\partial x^t} = \begin{array}{c|c|c|} & d_r & q_r \\ \hline a & -\sin\theta & \cos\theta \\ \hline b & -\cos\theta & -\sin\theta \\ \hline \end{array} \quad (\delta)$$

4.) C_σ^k is given in equ. 33.
5.)

$$\frac{\partial C_k^\delta}{\partial x^t} C_\sigma^k = \begin{array}{c|c|c|} & d_r & q_r \\ \hline d_r & 0 & 1 \\ \hline q_r & -1 & 0 \\ \hline \end{array} \quad (\delta)$$

6.) Multiplying it by equ. 39 (that is changing in equ. 39 the column d_r to $-q_r$, the column q_r to d_r and leaving out all other columns) *the result is identical with equ. 51.* That is with the new definition $\Gamma_{t\alpha,\sigma}$ has the same matrix that previously $\Gamma_{(t\alpha),\sigma}$ had.

XXVI. Summary of Voltage Vectors

Summarizing the *voltages* due to the presence of flux lines (defining $\Gamma_{t\alpha,\sigma}$ by equ. 91)

1.) $L_{\alpha\sigma}di^\alpha/dt$	$= d\varphi_\sigma/dt$	$= \ldots$	induced voltage due to the variation of currents.
2.) $[t\alpha, \sigma]\, i^\alpha i^t$	$= \phi_{t\sigma}i^t$	$= \ldots$	Coriolis voltage due to the motion of coordinate axes.
3.) $T_{t\alpha\sigma}i^\alpha i^t$	$= \psi_{t\sigma}i^t$	$= \ldots$	generated voltage due to the motion of rotor conductors.
4.) $\Gamma_{t\alpha,\sigma}i^\alpha i^t$	$= \bar{\psi}_{t\sigma}i^t$	$= \ldots$	total generated voltage.

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Summarizing the various *flux-density* vectors:

1.) $\psi_{\alpha t} = -\psi_{t\alpha} = \Gamma_{\alpha\gamma,t}i^\gamma = T_{\alpha\gamma t}i^\gamma$		$= \ldots$	rotor flux density.
2.) $\phi_{t,\alpha} = [t\gamma, \alpha]\, i^\gamma$		$= \ldots\ldots\ldots\ldots$	Coriolis flux density.
3.) $\bar{\psi}_{t,\alpha} = \psi_{t\alpha} + \phi_{t,\alpha}$	$= \Gamma_{t\gamma,\alpha}i^\gamma$	$= \ldots$	sum of the rotor and Coriolis flux densities.

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Summarizing the *induced voltages* (if $\varphi_\alpha = L_{\alpha\beta}i^\beta$ is the flux-linkage vector)

1.) $L_{\alpha\beta}di^\beta/dt$	$= d\bar{\varphi}_\alpha/dt$	$= \ldots\ldots$	induced voltage
2.) $(dL_{\alpha\beta}/dt)i^\beta$		$= \ldots\ldots\ldots\ldots$	Coriolis voltage
3.) $d(L_{\alpha\beta}i^\beta)/dt$	$= d\varphi_\alpha/dt$	$= \ldots\ldots$	sum of the induced and Coriolis voltages

. . . . 94

The Coriolis voltage may be considered either as an induced or as a generated voltage.

Hence the Equation of Voltage can be written in two different forms in terms of fluxes

$$\boxed{e_\alpha = R_{\alpha\beta}i^\beta + \frac{d\bar{\varphi}_\alpha}{dt} + \bar{\psi}_{t,\alpha}i^t} \quad \ldots\ldots\ldots\ldots\ldots\ldots 95$$

$$\boxed{e_\alpha = R_{\alpha\beta}i^\beta + \frac{d\varphi_\alpha}{dt} + \psi_{t\alpha}i^t} \quad \ldots\ldots\ldots\ldots\ldots\ldots 96$$

This latter equation reduces to the form given by Maxwell equ. 72 when the rotor coördinate axes are connected to the moving conductors since the last term drops out.

It should be especially noted that $i^\beta dL_{\alpha\beta}/dt$ can not be interpreted as the voltage due to the motion of the conductors. It is the voltage due to the motion of the coordinate axes, the Coriolis voltage.

XXVII. The Equation of Torque

The Equation of Torque can be set up by making in equ. 75 σ equal to t.

The value of $-\frac{1}{2}\partial L_{\alpha\gamma}/\partial x^t = [\gamma\alpha, t]$ is zero in machines with stationary axes. In machines with moving axes its value must be equal to $L_{mk}C^k_t \partial C^m_\gamma/\partial x^\alpha$ from the transformation formula of $\Gamma_{\gamma\alpha,\,t}$ (equ. 46) changing from stationary to moving axes, which however is zero. Hence in any coordinate system

$$[\gamma\alpha, t] = 0 \qquad \text{97}$$

(The Coriolis voltage $[t\alpha,\sigma]$ or $[\gamma t, \sigma]$ never reduces to zero in a machine with moving axes for any definition of $[\gamma\alpha, \sigma]$ by equs. 8 or 11 or 80 if equ. 46 is used since the denominator on the right hand side is x^t at least once, unlike for the torque.)

The value of $T_{\gamma\alpha t}$ is

$$T_{\gamma\alpha t} = -L_{\alpha\delta}C^n_\gamma \frac{\partial C^\delta_n}{\partial x^t} \qquad \text{98}$$

Hence the *Equation of Torque* is

$$e_t = R_{tt}i^t + L_{tt}\frac{di^t}{dt} + T_{\gamma\alpha t}i^\gamma i^\alpha \qquad \text{99}$$

That is

$$\Gamma_{\gamma\alpha,\,t} = T_{\gamma\alpha t} \qquad \text{100}$$

Since $T_{\alpha\gamma t}i^\gamma$ represents the resultant rotor flux-density vector $\psi_{\alpha t}$ equ. 93 *the torque of all machines is due to the interaction of the rotor currents and the rotor flux-density wave*, that is

$$\boxed{\text{Torque} = \psi_{\beta t}i^\beta} \qquad \text{101}$$

XXVIII. The Equation of Power

a.) If the shaft coordinate is excluded, the Equation of Power is found from the Equation of Voltage, equ. 90 by multiplying through with the current i^σ

$$e_\sigma i^\sigma = R_{\alpha\sigma} i^\alpha i^\sigma + L_{\alpha\sigma} \frac{di^\alpha}{dt} i^\sigma + [t\alpha, \sigma]\, i^t i^\alpha i^\sigma + T_{t\alpha\sigma} i^t i^\alpha i^\sigma \ldots\ldots 102$$

The time rate of change of the stored magnetic energy for machines with moving coordinate axes is (if α and β are stationary, m and n moving axes)

$$\frac{dT}{dt} = \frac{1}{2} \frac{d(L_{\alpha\beta} C^\alpha_m C^\beta_n i^m i^n)}{dt} = L_{mn} \frac{di^m}{dt} i^n + \frac{1}{2} \frac{dL_{\alpha\beta}}{dt} i^\alpha i^\beta + L_{\alpha\beta} \frac{\partial C^\alpha_m}{\partial x^t} i^m i^n i^t \ldots 103.$$

The last term is the power due to the Coriolis voltage, the previous term is zero and

$$\frac{dT}{dt} = L_{\alpha\sigma} \frac{di^\alpha}{dt} i^\sigma + [t\alpha, \sigma]\, i^t i^\alpha i^\sigma \ldots\ldots\ldots\ldots\ldots 104$$

$$e_\sigma i^\sigma = R_{\alpha\sigma} i^\alpha i^\sigma + \frac{dT}{dt} + \text{Mechanical Output} \ldots\ldots\ldots 105$$

where $$\text{Mechanical Output} = T_{t\alpha\sigma} i^t i^\alpha i^\sigma \ldots\ldots\ldots\ldots\ldots 106$$

Since the output is also equal torque times velocity it follows that the Coriolis flux-density does not contribute to the torque and the Coriolis voltage does not contribute to the mechanical output only to the variation of the stored magnetic energy.

b.) If the shaft coordinate is not excluded the Equation of Power is found by multiplying through equ. 75 by i^σ. Since $-\frac{1}{2}\partial L_{\alpha\gamma}/\partial x^\sigma$ is always zero

$$e_\sigma i^\sigma = R_{\alpha\sigma} i^\alpha i^\sigma + L_{\alpha\sigma} \frac{di^\alpha}{dt} i^\sigma + \frac{\partial L_{\alpha\sigma}}{\partial x^\gamma} i^\alpha i^\sigma i^\gamma \ldots\ldots\ldots\ldots 107$$

The other terms cancel. Since from equ. 104 the last two terms are equal to dT/dt, hence

$$e_\sigma i^\sigma = R_{\alpha\sigma} i^\alpha i^\sigma + \frac{dT}{dt} \ldots\ldots\ldots\ldots\ldots\ldots\ldots 108$$

XXIX. The Various Definitions of $\Gamma_{\alpha\beta,\gamma}$

a.) *It should be noted that there are several definitions of $\Gamma_{\alpha\beta,\gamma}$ each definition giving different matrices for $\Gamma_{t\beta,\gamma}$, $\Gamma_{\alpha t,\gamma}$ and $\Gamma_{\alpha\beta,t}$ but still the same final voltages and torques.*

In defining $\Gamma_{\alpha\beta,\gamma}$ from the *standard Lagrangian equation* (sections V and VI) there were four different definitions of $\Gamma_{\alpha\beta,\gamma}$ according to the definition of the Christoffel symbol with two (equ. 11) or with three terms (equ. 8) and according to the order of the indices in defining them as $\Gamma_{mn,k}$ or $\Gamma_{nm,k}$.

In defining $\Gamma_{\alpha\beta,\gamma}$ from the *generalized equation of Lagrange* (section XXIV) the same arbitrary definitions can be selected but even so the three matrices of $\Gamma_{\alpha\beta,\gamma}$ are different from those above.

1.) *The rotor generated voltage may be defined* as $\Gamma_{t\beta,\gamma}i^{\beta}$ (equ. 91) or $\Gamma_{\alpha t,\gamma}$ (equ. 51) *with matrix 51 or as the sum of two matrices 57 or 58* when defined as $\Gamma_{(t\beta),\gamma}$ in section VId.

2.) *The torque may be defined with matrix 51* as given in equ. 98 by $T_{\gamma\alpha t}$ *or with its symmetrical part, matrix 56*, since the torque is a homogeneous quadratic form and the skew-symmetrical matrix gives zero torque

3.) The *Coriolis voltage* may be defined as $[t\beta,\gamma]$ or $[\alpha t,\gamma]$ or $[(t\beta),\gamma]$ by equ. 8 or 11.

b.) Also it should be noted that the transformation formula 46 could have been defined also as

$$\Gamma_{\epsilon\sigma,\pi} = \Gamma_{nm,k}C^n_{\epsilon}C^m_{\sigma}C^k_{\pi} + L_{mk}\frac{\partial C^m_{\sigma}}{\partial x^{\epsilon}}C^k_{\pi} \qquad \text{109}$$

if equ. 11 had been defined as $\Gamma_{nm,k}$ instead of $\Gamma_{mn,k}$. The definition 46 was assumed to conform to the order of the indices as given by Schouten.

c.) *For purposes of quick calculations $\Gamma_{\alpha\beta,\gamma}$ for the representative machine with stationary coordinate axes should be defined as given by equ. 75 since that equation defines both $\Gamma_{t\alpha,\sigma}$ and $\Gamma_{\gamma\alpha,t}$ with the same matrix 51 and saves the calculation of different formulae for the torque and voltage. For any other machine $\Gamma_{\alpha\beta,\gamma}$ is found by a transformation of coordinates with the aid of equ. 109.*

XXX. Physical Definitions of the Dynamical Equations

a.) For the representative machine with moving coordinate axes *the Equation of Motion of Lagrange*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^k}\right) - \frac{\partial T}{\partial x^k} + \frac{\partial F}{\partial \dot{x}^k} = f_k \qquad \text{110}$$

can be divided into the Equation of Voltage

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^k}\right) + \frac{\partial F}{\partial \dot{x}^k} = e_k \ldots\ldots\ldots\ldots\ldots\ldots 111$$

where the first term of the left-hand member gives all the voltages due to the presence of flux lines. The Equation of Torque is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^t}\right) - \frac{\partial T}{\partial x^t} + \frac{\partial F}{\partial \dot{x}^t} = f_t \ldots\ldots\ldots\ldots\ldots\ldots 112$$

where the first term represents the time rate of change of angular momentum and the second term the machine torque.

b.) The physical interpretation of the *generalized Equation of Motion* valid for all electrical machinery

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\sigma}\right) - \frac{\partial T}{\partial x^\sigma} + \frac{\partial T}{\partial \dot{x}^\sigma}\left(\frac{\partial C_k^\delta}{\partial x^n} - \frac{\partial C_n^\delta}{\partial x^k}\right) C_\sigma^k C_\gamma^n \dot{x}^\gamma + \frac{\partial F}{\partial \dot{x}^\sigma} = f_\sigma \ldots\ldots 113$$

is different from that of the standard form. The *Equation of Voltage* is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\sigma}\right) + \frac{\partial T}{\partial \dot{x}^\delta}\frac{\partial C_k^\delta}{\partial x^n} C_\sigma^k C_\gamma^n \dot{x}^\gamma + \frac{\partial F}{\partial \dot{x}^\sigma} = e_\sigma \ldots\ldots\ldots\ldots 114$$

where the first term on the left-hand member represents now only the induced and Coriolis voltages while the second member represents the rotor generated voltages. The extra term appears however only in the equations of the axes stationary on the rotor. For all stator axes and for all axes moving with the rotor the generalized equation reduces to the standard Lagrangian form.

The *Equation of Torque* is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial x^t}\right) - \frac{\partial T}{\partial x^t} - \frac{\partial T}{\partial \dot{x}^\delta}\frac{\partial C_n^\delta}{\partial x^t} C_\gamma^n \dot{x}^\gamma + \frac{\partial F}{\partial \dot{x}^t} = f_t \ldots\ldots\ldots\ldots 115$$

The second term on the left-hand side is identically zero (equ. 97) and the machine torque is given now by the third term. It is interesting to note that *although axis t remaines unchanged still the original equation is not valid for it only the generalized one.*

Equ. 107 shows that *the power due to the additional terms in the generalized equation is zero.* Hence the additional terms may be called "gyroscopic forces."

c.) These interpretations are in conformity with the rule given by

Appell for non-holonomic dynamical systems of various orders. These rules are the following:

1.) For the coordinate axes that are changed the generalized equations apply.

2.) For the coordinate axes that are not changed but occur in the transformation tensor also the generalized equation applies.

3.) For the coordinate axes that are not changed and that do not occur in the transformation tensor the generalized equations reduce to the original Lagrangian equation.

THE ABSOLUTE CALCULUS

XXXI. The Generalized "Per-unit" or Contravariant Quantities

a.) In connection with the analysis of synchronous machinery a simplification is generally used to avoid the use of conversion factors. It is assumed that the stator current is *numerically equal* to its flux-linkages in the rotor, that is the rotor flux-linkage $i^{ds}M_d$ is replaced in analytical work by i^{ds} and $i^{qs}M_q$ by i^{qs}. As a consequence of this assumption the currents and voltages are replaced by other quantities, the so-called "per-unit" quantities. For instance the per-unit stator terminal voltage is $e_{sd}M_d/r_{sd}$ instead of e_{sd}, etc.

The disadvantage of this limited assumption is that:

1.) the symmetry of all scalar equations for the stator and rotor is destroyed.

2.) it cannot be used for other machines.

b.) *In the Absolute Calculus there is a fundamental process of simplification called the "raising (or lowering) of indices" which in case of rotating machinery is equivalent to a generalization of the per-unit concept.*

The resultant flux-linkages of a winding α due to all currents is represented by $L_{\alpha\beta}i^\beta$. *In order to represent the resultant flux-linkages of each winding by the current itself flowing in that winding every operator of the machine is multiplied by the inverse of the metric tensor $L_{\alpha\beta}$ that is by $L^{\alpha\beta}$.* If the Equation of Motion is multiplied through by $L^{\alpha\delta}$, then

$$L^{\alpha\delta}e_\alpha = L^{\alpha\delta}R_{\alpha\beta}i^\beta + \frac{di^\alpha}{dt} + L^{\alpha\delta}\Gamma_{\beta\gamma,\alpha}i^\beta i^\gamma \dots\dots\dots\dots 116$$

1.) $e_\alpha L^{\alpha\delta}$ is denoted by e^δ and is called the "contravariant voltage." It corresponds to the expression "per-unit voltage."

2.) $L^{\alpha\delta}R_{\alpha\beta}$ is denoted by $R^{\delta}_{\cdot\beta}$ and called the "mixed resistance tensor."

3.) $L^{\alpha\delta}\Gamma_{\beta\gamma,\alpha}$ is denoted by $\Gamma^{\delta}_{\beta\gamma}$ and called the generalized Christoffel symbol of the "second" kind. The notation $\Gamma_{\beta\gamma}^{\cdot\,\delta}$ is not used because $\Gamma_{\beta\gamma}^{\delta}$ is not a tensor. Its transformation formula is

$$\boxed{\Gamma_{\alpha\beta}^{\gamma} = \Gamma_{\delta\epsilon}^{\gamma} C_{\alpha}^{\delta} C_{\beta}^{\epsilon} C_{\sigma}^{\gamma} + \frac{\partial C_{\alpha}^{\sigma}}{\partial x^{\beta}} C_{\sigma}^{\gamma}} \qquad \text{.............. 117}$$

The Equation of Motion becomes in terms of contravariant quantities

$$\boxed{e^{\alpha} = R^{\alpha}_{\cdot\beta} i^{\beta} + \frac{di^{\alpha}}{dt} + \Gamma_{\beta\gamma}^{\alpha} i^{\beta} i^{\gamma}} \qquad \text{............... 118}$$

$$\boxed{e^{\alpha} = R^{\alpha}_{\cdot\beta} i^{\beta} + \frac{di^{\alpha}}{dt} + \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} i^{\beta} i^{\gamma} + T_{\beta\gamma}^{\cdot\cdot\,\alpha} i^{\beta} i^{\gamma}} \qquad \text{......... 119}$$

where $\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix}$ is the ordinary Christoffel symbol of the "second" kind. *It can be seen that $L_{\alpha\beta} di^{\alpha}/dt$ is replaced by di^{α}/dt in all equations* and:

1.) the symmetry of all equations is maintained,

2.) the simplification can be used for all rotating machines simply by multiplying all their operators by the inverse of the metric tensor $L_{\alpha\beta}$. In calculating the contravariant operators of a machine whose matrix of transformation is not square, first its covariant operators $L_{\alpha\beta}$, $\Gamma_{\beta\gamma,\alpha}$ etc. should be calculated and then only its contravariant operators $\Gamma_{\beta\gamma}^{\alpha}$, $R^{\alpha}_{\cdot\beta}$ etc. (See section XXXIV.)

In the accompanying Table IV the calculated values of the contravariant operators of both representative machines are given.

c.) An important physical interpretation can be given to the inverse of the metric tensor. Let $\bar{L}_{sd}$ denote the inductance of the stator winding along the direct axis while the rotor winding is short-circuited. Then $A = L_{sd}L_{rd} - M_d^2 = L_{rd}\bar{L}_{sd}$. If $M_d/L_{rd} = v_{rd}$ then $L^{\alpha\beta}$ also can be written as

$$L^{\alpha\beta} = \begin{array}{c|cccc} & d_s & d_r & q_r & q_s \\ \hline d_s & 1/\bar{L}_{sd} & -v_{rd}/\bar{L}_{sd} & 0 & 0 \\ d_r & -v_{rd}/\bar{L}_{sd} & 1/\bar{L}_{rd} & 0 & 0 \\ q_r & 0 & 0 & 1/\bar{L}_{rq} & -v_{rq}/\bar{L}_{sq} \\ q_s & 0 & 0 & -v_{rq}/\bar{L}_{sq} & 1/\bar{L}_{sq} \end{array} \qquad \text{...120}$$

The equation $e_{\alpha}L^{\alpha\beta} = e^{\beta} = i^{\beta}$ represents the contravariant impressed voltages as currents in the various axes due to covariant voltages

Let $A = L_{sd}L_{rd} - (M_d)^2$ and $B = L_{sq}L_{rq} - (M_q)^2$

$L^{mn} =$

$m \backslash n$	d_s	a	b	q_s	
d_s	L_{rd}/A	$-M_d \cos\theta/A$	$M_d \sin\theta/A$	0	0
a	$-M_d\cos\theta/A$	$L_{sd}\cos^2\theta/A + L_{sq}\sin^2\theta/B$	$(L_{sq}/B - L_{sd}/A)\sin\theta\cos\theta$	$-M_q\sin\theta/B$	0
b	$M_d\sin\theta/A$	$(L_{sq}/B - L_{sd}/A)\sin\theta\cos\theta$	$L_{sd}\sin^2\theta/A + L_{sq}\cos^2\theta/B$	$-M_q\cos\theta/B$	0
q_s	0	$-M_q\sin\theta/B$	$-M_q\cos\theta/B$	L_{rq}/B	0
t	0	0	0	0	$1/L$

$R^{m}_{\cdot m} =$

$m \backslash n$	d_s	a	b	q_s	t
d_s	$r_{sd}L_{rd}$ A	$-r_r M_d \cos\theta/A$	$-M_d L_{sq}\cos\theta/B$	0	0
a	$-r_{sd}M_d\cos\theta/A$	$r_r(L_{sd}\cos^2\theta + L_{sq}\sin^2\theta)/B$	$r_r(L_{sq}/B - L_{sd}/A)\sin\theta\cos\theta$	$-r_{sq}M_q\sin\theta/B$	0
b	$r_{sd}M_d\sin\theta/A$	$r_r(L_{sq}/B - L_{sd}/A)\sin\theta\cos\theta$	$r_r(L_{sd}\sin^2\theta/A + L_{sq}\cos^2\theta)/B$	$-r_{sq}M_q\cos\theta/B$	0
q_s	0	$-r_r M_q \sin\theta/B$	$-r_r M_q\cos\theta/B$	$r_{sq}L_{rq}/B$	0
t	0	0	0	0	r/L

$\Gamma^{n}_{mt} =$

$m \backslash n$	d_s	a	b	q_s
d	0	$-M_d L_{sq}\sin\theta/B$	$-M_q L_{sq}\cos\theta/B$	$M_d M_q/B$
a	$\dfrac{-M_d L_{rq}\sin\theta}{A}$	$\sin\theta\cos\theta[M_d^2/A - M_q^2/B + (L_{rq} - L_{rd})(L_{sd}/A + L_{sq}/B)]$	$-M_d^2\sin^2\theta/A + M_q^2\cos^2\theta/B + (L_{rq} - L_{rd})(L_{sq}\cos\theta/B - L_{sd}\sin^2\theta/A)$	$\dfrac{M_q L_{rd}\cos\theta}{B}$
b	$\dfrac{-M_d L_{rq}\cos\theta}{A}$	$M_d^2\cos^2\theta/A + M_q^2\sin^2\theta/B + (L_{rq} - L_{rd})(L_{sd}\cos^2\theta/A - L_{sq}\sin^2\theta/B)$	$\sin\theta\cos\theta[M_q^2/B - M_d^2/A - (L_{rq} - L_{rd})(L_{sq}/B + L_{sd}/A)]$	$\dfrac{-M_q L_{rd}\sin\theta}{B}$
q_s	$-M_d M_q/A$	$M_q L_{sd}\cos\theta/A$	$-M_q L_{sd}\sin\theta/A$	0

$\Gamma^{t}_{mn} =$

$m \backslash n$	d_s	a	b	q_s
d_s	0	$M_d\sin\theta/2L$	$M_d\cos\theta/2L$	0
a	$M_d\sin\theta/2L$	$(L_{rd} - L_{rq})\sin\theta\cos\theta/L$	$(L_{rd} - L_{rq})(\cos^2\theta - \sin^2\theta)/2L$	$-M_q\cos\theta/2L$
b	$M_d\cos\theta/2L$	$(L_{rd} - L_{rq})(\cos^2\theta - \sin^2\theta)/2L$	$(L_{rq} - L_{rd})\sin\theta\cos\theta/L$	$M_q\sin\theta/2L$
q_s	0	$-M_q\cos\theta/2L$	$M_q\sin\theta/2L$	0

THE CONTRAVARIANT OPERATORS OF THE REPRESENTATIVE MACHINE WITH MOVING COORDINATE AXES

$L^{\pi\sigma} =$

$\pi \backslash \sigma$	d_s	d_r	q_r	q_s	t
d_s	L_{rd}/A	$-M_d/A$	0	0	0
d_r	$-M_d/A$	L_{sd}/A	0	0	0
q_r	0	0	L_{sq}/B	$-M_q/B$	0
q_s	0	0	$-M_q/B$	L_{rq}/B	0
t	0	0	0	0	$1/L$

$R^{\pi}_{\cdot\sigma} =$

$\pi \backslash \sigma$	d_s	d_r	q_r	q_s	t
d_s	$r_{sd}L_{rd}/A$	$-r_r M_d/A$	0	0	0
d_r	$-r_{sd}M_d/A$	$r_r L_{sd}/A$	0	0	0
q_r	0	0	$r_r L_{sq}/B$	$-r_{sq}M_q/B$	0
q_s	0	0	$-r_r M_q/B$	$r_{sq}L_{rq}/B$	0
t	0	0	0	0	r/L

$\Gamma^{\pi}_{\varepsilon t} =$

$\varepsilon \backslash \pi$	d_s	d_r	q_r	q_s
d_s	0	0	$-M_d L_{sq}/B$	$M_d M_q/B$
d_r	0	0	$-L_{rd}L_{sq}/B$	$M_q L_{rd}/B$
q_r	$-M_d L_{rq}/A$	$L_{rq}L_{sd}/A$	0	0
q_s	$-M_d M_q/A$	$M_q L_{sd}/A$	0	0

$\Gamma^{t}_{\varepsilon\pi} =$

$\varepsilon \backslash \pi$	d_s	d_r	q_r	q_s
d_s	0	0	$M_d/2L$	0
d_r	0	0	$(L_{rd} - L_{rq})/2L$	$-M_q/2L$
q_r	$M_d/2L$	$(L_{rd} - L_{rq})/2L$	0	0
q_s	0	$-M_q/2L$	0	0

THE CONTRAVARIANT OPERATORS OF THE REPRESENTATIVE MACHINE WITH STATIONARY COORDINATE AXES

TABLE IV.

impressed while all axes are short-circuited. Also while *the metric tensor is the measure of the permeances of the various magnetic circuits with the windings open-circuited, the inverse metric tensor in the measure of the reluctances of the magnetic circuits with the windings short-circuited.* In the first case the magnetic lines follow paths with the maximum possible permeances, in the second case they follow the paths with the minimum possible permeances.

d.) *Each term of the inverse of a dyadic as $L^{\alpha\beta}$ is found as follows:*

1.) Replace each term by the determinant formed when the corresponding row and column are removed.

2.) Multiply it by plus or minus one according as the term occupies an even or odd element counted from the upper left hand corner.

3.) Interchange the two indices.

4.) Divide it by the determinant of the dyadic.

XXXII. Absolute or Covariant Differentiation

a.) It has been shown in section XIII that the derivative or the differential of a tensor is *not* a tensor. In textbooks on the Absolute Calculus a new type of differentiation is introduced, called the "absolute" or "covariant" or "intrinsic" differentiation, denoted by δ which:

1.) *always produces a tensor from a tensor* as a result of the differentiation that is it determines another invariant from a known invariant by a routine process

2.) also obeys all the rules of ordinary differentiation, among others the following rule $\delta(AB) = (\delta A)B + A(\delta B)$

3.) as a special case reduces to ordinary differentiation.

The absolute differentials of tensors of rank one and two are:

$$\boxed{\delta A^\alpha = dA^\alpha + \Gamma^\alpha_{\beta\gamma} A^\beta dx^\gamma} \qquad (121)$$

$$\delta A_\alpha = dA_\alpha - \Gamma^\gamma_{\alpha\beta} A_\gamma dx^\beta \qquad (122)$$

$$\delta A^{\alpha\beta} = dA^{\alpha\beta} + \Gamma^\alpha_{\gamma\delta} A^{\gamma\beta} dx^\delta + \Gamma^\beta_{\gamma\delta} A^{\alpha\gamma} dx^\delta \qquad (123)$$

$$\delta A_{\alpha\beta} = dA_{\alpha\beta} - \Gamma^\gamma_{\alpha\delta} A_{\beta\gamma} dx^\delta - \Gamma^\gamma_{\beta\delta} A_{\alpha\gamma} dx^\delta \qquad (124)$$

b.) The absolute derivative of i^α with respect to time is

$$\frac{\delta i^\alpha}{\delta t} = \frac{di^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} i^\beta \frac{dx^\gamma}{dt} = \frac{di^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} i^\beta i^\gamma \qquad (125)$$

If this equation is substituted into the two forms of the Equation of Motion the contravariant form becomes

$$\boxed{e^\alpha = R^\alpha{}_\beta i^\beta + \frac{\delta i^\alpha}{\delta t}} \qquad \ldots\ldots 126$$

The covariant form becomes

$$\boxed{e_\alpha = R_{\alpha\beta} i^\beta + L_{\alpha\beta}\frac{\delta i^\beta}{\delta t}} \qquad \ldots\ldots 127$$

Each term in these equations is a tensor and they express the fact that *the theory of rotating machinery during acceleration is identical with that of stationary networks with resistances and inductances provided ordinary differentiation is replaced by absolute differentiation.*

The tensor form of the Equation of Voltage of Maxwell is

$$\boxed{e_\alpha = R_{\alpha\beta} i^\beta + \frac{\delta \varphi_\alpha}{\delta t}} \qquad \ldots\ldots 128$$

$L_{\alpha\beta}\delta i^\beta/\delta t$ also $\delta\varphi_\alpha/\delta t$ represent all the voltages due to the presence of flux-lines, that is the sum of the induced, generated and Coriolis voltages.

c.) *The absolute derivative of the metric tensor is zero.* That is by equs. 124, 82 and 8

$$\frac{\delta L_{\alpha\beta}}{\delta x^\delta} = \frac{\partial L_{\alpha\beta}}{\partial x^\delta} - \Gamma^\gamma_{\alpha\delta} L_{\beta\gamma} - \Gamma^\gamma_{\beta\delta} L_{\alpha\gamma} = \frac{\partial L_{\alpha\beta}}{\partial x^\delta} - \Gamma_{\alpha\delta,\beta} - \Gamma_{\beta\delta,\alpha} = \frac{\partial L_{\alpha\beta}}{\partial x^\delta}$$

$$- [\alpha\delta,\beta] - [\beta\delta,\alpha] - T_{\alpha\delta\beta} - T_{\beta\delta\alpha} = 0 \qquad \ldots\ldots 129$$

d.) Since the sum of $L_{\alpha\beta}di^\beta/dt$ and $\Gamma_{\beta\gamma,\alpha}i^\beta i^\gamma$ is a tensor, in calculating the performance of any particular rotating machine it is not necessary to use their transformation formulae in equs. 44 and 46. *As long as it is not necessary to calculate separately the induced and generated voltages, it is sufficient to treat each expression in the Equation of Motion as a tensor* and use the simple formula of tensor transformations (equ. 36). This represents large savings of routine labor.

If the rotating machine has no moving coordinate axes, then all expressions are transformed as tensors.

$\Gamma_{\alpha\beta,t}$ *and the expression for torque* $\Gamma_{\alpha\beta,t}i^\alpha i^t$ *are transformed in all cases as tensors.*

THE THEORY OF NON-EUCLIDEAN SPACES

XXXIII. Types of Spaces

a.) The theory of a set of linear differential equations is usually analyzed in the language of multidimensional geometry. The variables are assumed to form a curvilinear coordinate system in an n-dimensional space or manifold and the equations represent lines, two-, three- etc. dimensional surfaces curved in an n-dimensional space.

By denying the fifth postulate of Euclid that *only one* line can be drawn through a point parallel to a given line, *two consistent geometries* have been set up during the last century, after two thousand years of unsuccessful efforts of proving it. By assuming that *more than one parallel line* are possible Lobachevsky, Bolyai, etc. built up the "hyperbolic geometry" and by assuming *that no parallel lines* are possible Gauss, Riemann, Klein, etc. built up the "elliptic geometry." Both geometries give as special cases the Euclidean or "parabolic geometry." It seems that physical phenomena can be described only in terms of motions in a space whose geometry is elliptic. It is the geometry of the various types of elliptic spaces that will be considered below.

b.) *The various types of spaces differ from each other in the definition of two all important concepts. These concepts are the "metric" and the "connection."*

1.) *To compare two vectors located in one point* along different directions an infinitesimal length ds (called the "line element") has to be defined. If ds^2 is defined in every coordinate system by the homogeneous quadratic form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \qquad \text{.....................130}$$

where $g_{\alpha\beta}$ is a symmetric tensor of rank two (see equ. 3) the space is called a "metric space," $g_{\alpha\beta}$ the "metric tensor" and ds^2 the "metric." (The formula is a generalization of the Pythagorean Theorem for n infinitesimal oblique axes.) If ds^2 is defined by a different function of dx^α the space is sometimes called "Finsler space."

2.) *To compare two vectors located at different points* of the space first it is necessary to shift one "parallel to itself" to the other. To visualize the problem let two equal vectors lie at different points on the Earth's surface. If both lie in north-south direction, viewed from the Earth's surface the difference between them δA^α may *arbitrarily* be assumed to be zero (after shifting one parallel to itself along a great circle) or any other quantity. But viewed from the outside space the difference between them is something else, dA, a *definite* quantity. The relation between the two differences is

$$\delta A^\alpha = dA^\alpha + \Gamma^\alpha_{\beta\gamma} A^\beta \, dx^\gamma \ldots\ldots\ldots\ldots\ldots\ldots 131$$

where $\Gamma^\alpha_{\beta\gamma}$ *is any arbitrary number system of rank three* (see equs. 15 and 16). Each coefficient is a function of the variables dx^α and is called the "coefficient of connection."

c.) *If* $\Gamma^\alpha_{\beta\gamma}$ *is defined in terms of the metric tensor as*

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\beta\gamma}}{\partial x^\gamma} + \frac{\partial g_{\gamma\delta}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right) g^{\alpha\delta} = \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} \ldots\ldots\ldots 132$$

the space is called "Riemannian" (see equ. 8) *otherwise "non Riemannian" or "affine"* (see equ. 80). In a non-Riemannian space a metric may or may not be defined, also $\Gamma^\alpha_{\beta\gamma}$ may be symmetrical in β and γ (so that $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$) or asymmetrical. *A special case of the asymmetric affine space is the "space with torsion"* (equ. 129). *All rotating electrical machinery are equivalent to such spaces.* A special case of the symmetrical affine space is the "Weyl space" and a special case of the Riemannian space is the "flat" space for which the curvature tensor $K_{\alpha\beta\gamma\delta}$, defined later in equ. 157, is zero. If the line element is definite that is if it is always different from zero the flat space is called "Euclidean." If in a Euclidean space the axes are oblique instead of curvilinear each coefficient of the Christoffel symbol is zero (equ. 136). If the axes are orthogonal the metric tensor reduces to a unit matrix.

d.) Among the spaces the only ones that can be visualized are;

1.) The one-, two- and three-dimensional Euclidean space.

2.) The two-dimensional Riemannian space (the various types of surfaces).

XXXIV. Holonomic Sub-spaces

a.) It is important to note that the properties of a space are independent of the particular coordinate axes from which measurements are made. In other words *two spaces are equivalent if it is possible to pass from one space to the other and back by a transformation tensor* $C^\beta_\alpha = \partial x^\beta/\partial x^\alpha$ *and its inverse. If however* C^β_α *can not be expressed as* $\partial x^\beta/\partial x^\alpha$ *then no one-to-one relation exists between the points of the two spaces (since no equation can be set up between the two sets of variables* x^β *and* x^α*) and the two spaces are not equivalent.* Equivalent spaces are for instance the shunt polyphase commutator motor, fig. 4 and the induction motor, fig. 2.

b.) *If the matrix of the transformation tensor* $C^{\pi}_{\alpha} = \partial x^{\pi}/\partial x^{\alpha}$ *is not square* as above, *but rectangular, then the space which has less coordinate axes is said to be a "sub-space" of the other.* For instance the repulsion motor, fig. 6 is a sub-space of the induction motor, fig 2 or of the Déry motor fig. 7.

Among the possible sub-spaces the only ones visualizable are:

1.) The plane as a two-dimensional Euclidean sub-space in a three-dimensional Euclidean space.

2.) A surface like the sphere, as a two-dimensional Riemannian sub-space in a three-dimensional Euclidean space.

c.) If in the transformation tensor C^{π}_{α} the number of the old coordinate axes is n and the number of the new coordinate axes is m then *all the polyadics of the old space that have only covariant (lower) indices can be assigned to the sub-space by transforming them with the aid of the formulae developed in the previous pages.* For instance the metric tensor $L_{\alpha\beta}$ of the sub-space can be and should be defined as

$$L_{\alpha\beta} = L_{\pi\sigma}\, C^{\pi}_{\alpha} C^{\sigma}_{\beta} \qquad \text{.....................133}$$

where the dummy indices π and σ assume all values from one to n while the free indices α and β assume the values from one to m. The same variation of indices applies in the transformation of $\Gamma_{\pi\sigma,\epsilon}$. All these polyadics of the sub-space are called "induced" or "derived" polyadics.

Since the performance of many rotating machines is expressed in terms of induced polyadics, their performance may be considered as the motion of a particle describing a traiectory in an n-dimensional non-Riemannian space but constrained to move upon an m-dimensional non-Riemannian surface or sub-space.

d.) If it is attempted to transform a polyadic which has a contravariant or upper index, say $\Gamma^{\pi}_{\sigma\epsilon}$ the value of the inverse of C^{π}_{α} is required. But the matrix of C^{π}_{α} is rectangular and its inverse can not be calculated without a definition.

In case of non-Riemannian spaces for which no metric is defined it is customary to form a square matrix from the rectangular matrix by assuming additional coordinate axes outside the sub-space in the enveloping space with n-m dimensions. For instance an additional set of brushes may be put on the rotor of the repulsion motor, fig. 6 (and a winding on the stator quadrature axis) but they would remain permanently open-circuited. Then the new transformation tensor of the repulsion motor would be similar or identical with that of the shunt polyphase commutator motor, fig. 4, it would have an inverse and all "induced" contravariant polyadics could be calculated. Their value however depends on the position of the additional axes.

e.) In electrical machinery the representative machine has a metric tensor $L_{\pi\sigma}$ and so the sub-space has an induced metric. *It seems obvious to define derived contravariant polyadics for the sub-space simply by raising indices with the aid of* $L^{\alpha\beta}$. That is $\Gamma^{\gamma}_{\alpha\beta}$ for the sub-space is defined as $\Gamma_{\alpha\beta,\delta}L^{\delta\gamma}$.

The above definition of $\Gamma^{\gamma}_{\alpha\beta}$ may be taken also as the definition of the inverse connection tensor C^{α}_{π} corresponding to the formula

$$\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\epsilon}_{\pi\sigma}\, C^{\pi}_{\alpha}C^{\sigma}_{\beta}C^{\gamma}_{\epsilon} = (\Gamma_{\pi\sigma,\mu}\, C^{\pi}_{\alpha}C^{\sigma}_{\beta}C^{\mu}_{\delta})\, L^{\delta\gamma} = (\Gamma^{\epsilon}_{\pi\sigma}\, L_{\epsilon\mu})\, C^{\pi}_{\alpha}C^{\sigma}_{\beta}C^{\mu}_{\delta}L^{\delta\gamma}$$

which shows that the above definition of $\Gamma^{\gamma}_{\alpha\beta}$ is equivalent to the following definition of the inverse connection tensor

$$C^{\gamma}_{\epsilon} = C^{\mu}_{\delta}L_{\mu\epsilon}L^{\delta\gamma} \quad \ldots\ldots 134$$

C^{γ}_{ϵ} can be taken as the inverse of C^{ϵ}_{γ} since

$$C^{\gamma}_{\epsilon}C^{\epsilon}_{\alpha} = C^{\epsilon}_{\alpha}C^{\mu}_{\delta}\, L_{\mu\epsilon}L^{\delta\gamma} = L_{\alpha\delta}L^{\delta\gamma} = \delta^{\gamma}_{\epsilon} \quad \ldots\ldots 135$$

where $\delta^{\gamma}_{\epsilon}$ is a unit matrix with m unit coefficients in the diagonal.

XXXV. Non-holonomic Sub-spaces

In many cases in which the number of new coordinates is less than the number of old coordinates the transformation tensor C^{π}_{α} can not be written as $\partial x^{\pi}/\partial x^{\alpha}$. Such a case occurs for instance in the single-phase alternator fig. 3 where axis b should be removed. (In general moving coordinate axes form non-integrable transformation tensors.) Lately the expression "non-holonomic sub-space" has been introduced to denote the restricted positions of the particle.

For both types of sub-spaces with each point of the n-dimensional space there is associated an m-dimensional infinitesimal surface element and the particle must leave that point along the surface element. But *while in a true sub-space the surface elements may be combined to form a family of surfaces so that the particle can never leave the particular surface it happens to be on, in a non-holonomic sub-space the surface elements can not be combined into a surface and the particle may pass from any one point of the n-dimensional space to any of its other points.*

To visualize the problem, in a repulsion motor, fig. 6 which is a true two-dimensional sub-space in a three- (or four-) dimensional non-Riemannian space no current (or rather no charges) can flow along axis b, at right angles to axis a. In a single-phase alternator where the axis a is rotating no axis can be assigned in which no current (no charges) could flow. Hence the single-phase alternator is a two- (or three-) dimensional non-holonomic sub-space of a four-dimensional non-Riemannian space, that is of the representative machine with stationary coordinates. (It happens that it may also be considered as a holonomic sub-space of a four-dimensional Riemannian space, the polyphase alternator, or also as an independent three-dimensional Riemannian space.

It should be remembered however that *in case of electrical machinery* the enveloping n-dimensional space need not be considered even in case of non-holonomic sub-spaces. *Every m-dimensional sub-space, holonomic or non-holonomic, is in its own right an m-dimensional non-Riemannian space since the new coordinates, even though they are non-holonomic, can be calculated immediately from the dynamical equations.*

XXXVI. Spaces with Infinite Dimensions—Hilbert Spaces

a.) If the number of the new coordinate axes m is larger than the number of the old coordinate axes n, for instance in case of the frequency converter fig. 12, than the same equations of transformations may be used as previously with the tacit understanding that certain quantities are neglected.

To visualize the problem if three sets of axes at 120 degrees apart exist on the armature of a three-phase salient-pole synchronous machine instead of two at right angles then zero phase-sequence waves exist inside the machine (which may be considered as third-harmonic space waves). However the metric tensor of the representative machine given in equ. 3 does not contain zero phase-sequence reactances, hence the final equations give only a first approximation to the actual phenomena. A second approximation could have been made if the representative machine itself would have had at least three coordinate axes on each rotor layer, since then zero phase-sequence reactances could have been incorporated in the metric tensor.

That is *a more general representative machine has more than two axes on each winding* and the representative machine of section IV is a sub-space of this more general representative machine.

b.) In the representative machines of this paper there are two axes of magnetic symmetry assumed on the stator. In actual machines each stator and rotor tooth and slot is also an axis of magnetic symmetry, hence to each of them a coordinate axis may be assigned.

Since the windings are distributed in slots the magnetomotive force wave introduces additional harmonic waves in unbalanced phase windings etc. so that *the most general representative machine must have an infinite number of coordinate axes representing a space with infinite number of dimensions*, some axes being stationary, some moving with the rotor and some rotating with odd speeds. Such cases have practical importance especially in squirrel-cage induction motors of smaller sizes where the harmonics are known to distort the ideal performance beyond recognition in many cases. It is intended to undertake the systematic dynamical treatment of such generalized systems with infinite coordinate axes in some future paper.

XXXVII. Euclidean Spaces

a.) If the *speed* of the rotor is assumed to be *constant* the Equation of Voltage (equ. 17) can also be written as

$$e_\alpha = Z_{\alpha\beta} i^\beta = (R_{\alpha\beta} + \Gamma_{\beta t,\alpha} i^t)\, i^\beta + L_{\alpha\beta} \frac{di^\beta}{dt} \quad \ldots\ldots\ldots\ldots 136$$

where the terms in parenthesis are equivalent to resistances. Since the coefficients of connection are zero, *the sudden short-circuit performance of rotating machines with the speed maintained constant can be represented by the motion of a particle in an n-dimensional Euclidean space with oblique cartesian coordinate axes just as if the rotating machines were stationary networks.*

b.) If the rotor is assumed to be *stationary* then in the above equation $\Gamma_{\beta t,\alpha} i^t i^\beta$ is zero and *the transient impedance tensor $Z_{\alpha\beta}$ becomes symmetrical.* Such stationary electrical apparatus are the transformer, induction regulator etc.

The transient impedance tensor of the single phase transformer is

$$Z_{\alpha\beta} = \begin{array}{c|c|c|} & p & s \\ \hline p & r_1 + L_1 p & Mp \\ \hline s & Mp & r_2 + L_2 p \\ \hline \end{array}$$

The transient impedance tensor of the three-circuit transformer is

$$Z_{\alpha\beta} = \begin{array}{c|c|c|c|} & a & b & c \\ \hline a & r_a + L_{aa} p & M_{ab} p & M_{ac} p \\ \hline b & M_{ba} p & r_b + L_{bb} p & M_{bc} p \\ \hline c & M_{ca} p & M_{cb} p & r_c + L_{cc} p \\ \hline \end{array}$$

Similar matrix structures can be set up for other stationary electrical apparatus also, such as vacuum tubes etc.

When resistances, inductances, transformers, vacuum tubes etc. are connected with rotating machines in any manner whatever each unit is considered as a rotating machine or better said each rotating machine is considered as a simple impedance.

c.) With steady *a-c* voltages impressed at the terminals the variable i^α assumes complex values in each axis. The study of such spaces is outside the scope of this paper.

HUNTING

XXXVIII. The Equation of Small Oscillations

Let an *infinitesimal disturbance* de^α (infinitesimal terminal voltage or shaft torque) be applied to any machine while it is either accelerated or is in a uniform motion. Then the Equation of Motion (equ. 126)

$$e^\alpha = R^\alpha_{.\beta} i^\beta + \frac{di^\alpha}{dt} \Gamma^\alpha_{\beta\gamma} i^\beta i^\gamma$$

changes to

$$e^\alpha + de^\alpha + \frac{\partial e^\alpha}{\partial x^\beta} dx^\beta = (R^\alpha_{.\beta} + dR^\alpha_{.\beta})\,(i^\beta + di^\beta) + \frac{d(i^\alpha + di^\alpha)}{dt}$$
$$+ (\Gamma^\alpha_{\beta\gamma} + d\Gamma^\alpha_{\beta\gamma})\,(i^\beta + di^\beta)\,(i^\gamma + di^\gamma).$$

If the Equation of Motion is subtracted from it and all products of infinitesimals are cancelled, the Equation of Small Oscillations in contravariant form is

$$\boxed{de^\alpha + \frac{\partial e^\alpha}{\partial x^\beta} dx^\beta = R^\alpha_{.\beta} di^\beta + dR^\alpha_{.\beta} i^\beta + \frac{d(di^\alpha)}{dt} + \Gamma^\alpha_{\beta\gamma} di^\beta i^\gamma + \Gamma^\alpha_{\beta\gamma} i^\beta di^\gamma + d\Gamma^\alpha_{\beta\gamma} i^\beta i^\gamma} \quad .137$$

In covariant form it is

$$\boxed{de_\alpha + \frac{\partial e_\alpha}{\partial x^\beta} dx^\beta = R_{\alpha\beta} di^\beta + L_{\alpha\beta}\frac{d(di^\beta)}{dt} + dL_{\alpha\beta}\frac{di^\beta}{dt} + \Gamma_{\beta\gamma.\alpha} di^\beta i^\gamma + \Gamma_{\beta\gamma.\alpha} i^\beta di^\gamma + d\Gamma_{\beta\gamma.\alpha} i^\beta i^\gamma} \quad ..138$$

Equ. 137 is equivalent to

$$de^\alpha = d(R^\alpha_{.\beta}\, i^\beta) + d\left(\frac{di^\alpha}{dt}\right) + d(\Gamma^\alpha_{\beta\gamma}\, i^\beta i^\gamma) \ldots\ldots\ldots\ldots 139$$

or to

$$de^{\alpha} = d(R^{\alpha}{}_{\beta}\, i^{\beta}) + d\left(\frac{\delta i^{\alpha}}{\delta t}\right) \qquad 140$$

XXXIX. Physical Interpretation

a.) The Equation of Small Oscillations also can be divided into two component equations, the *Equation of Small Voltages* by allowing the free index α assume any value but t. (The definitions of section VI a, b, c are followed)

$$de_{\alpha} + \frac{\partial e_{\alpha}}{\partial x^{\beta}}\, dx^{\beta} = R_{\alpha\beta}\, di^{\beta} + L_{\alpha\beta}\, \frac{d(di^{\beta})}{dt} + dL_{\alpha\beta}\, \frac{di^{\beta}}{dt} + \Gamma_{\beta t,\, \alpha}\, di^{\beta}\, i^{t}$$
$$+ \Gamma_{\beta t,\, \alpha}\, i^{\beta}\, di^{t} + d\Gamma_{\beta t,\, \alpha}\, i^{\beta}\, i^{t} \qquad 141$$

In terms of physical space vectors the equation becomes

$$de_{\alpha} + \frac{\partial e_{\alpha}}{\partial x^{\beta}}\, dx^{\beta} = R_{\alpha\beta}\, di^{\beta} + d\,\frac{d\varphi_{\alpha}}{dt} + d\psi_{t,\, \alpha}\, i^{t} + \psi_{t,\, \alpha}\, di^{t} \qquad 142$$

The *Equation of Small Torques* is found by allowing α assume the value t only

$$de_{t} = R_{tt}\, di^{t} + L_{tt}\, \frac{d(di^{t})}{dt} + \Gamma_{\beta\gamma,\, t}\, di^{\beta}\, i^{\gamma} + \Gamma_{\beta\gamma,\, t}\, i^{\beta}\, di^{\gamma} + d\Gamma_{\beta\gamma,\, t}\, i^{\beta}\, i^{\gamma} \qquad 143$$

In terms of space vectors the equation is

$$de_{t} = R_{tt}\, di^{t} + L_{tt}\, \frac{d(di^{t})}{dt} + d\psi_{\gamma,\, t}\, i^{\gamma} + \psi_{\gamma,\, t}\, di^{\gamma} \qquad 144$$

b.) *At the instant of disturbance two sets of vectors can be differentiated inside the machine:*

1.) the original vectors that exist before the disturbance satisfying the Equation of Motion (equs. 34 and 35) and are given in section XI

2.) a set of infinitesimal vectors representing changes in each of the above vectors.

c.) Some of the vector changes can be divided into two components. *Among the voltages:*

A.) the change in the generated voltage vector can be divided into

1.) $\psi_{t,\alpha} di^{t} = \Gamma_{\beta t,\alpha} i^{\beta} di^{t}$ due to the change of speed and the original flux density $\psi_{t,\alpha}$

2.) $d\psi_{t,\alpha} i^{t} = \Gamma_{\beta t,\alpha} di^{\beta} i^{t} + d\Gamma_{\beta t,\alpha} i^{\beta} i^{t}$ due to change of $\psi_{t,\alpha}$ and to original i^{t}

B.) the change in the torque vector can be divided into:

1.) $\psi_{\gamma,t} di^{\gamma} = \Gamma_{\beta\gamma,t} i^{\beta} di^{\gamma}$ due to change of current and to original $\psi_{\gamma,t}$

2.) $d\psi_{\gamma,t}i^\gamma = \Gamma_{\beta\gamma,t}di^\beta i^\gamma + d\Gamma_{\beta\gamma,t}i^\beta i^\gamma$ due to change of $\psi_{\gamma,t}$ and to original current.

Among the fluxes:

A.) The change in the flux-linkage vector $d\varphi_\alpha = d(L_{\alpha\beta}i^\beta)$ can be divided into $i^\beta dL_{\alpha\beta}$ and $L_{\alpha\beta}di^\beta$

B.) the change in the rotor flux-density vector $d\psi_{\gamma,t} = d(\Gamma_{\beta\gamma,t}i^\beta)$ can be divided into $d\Gamma_{\beta\gamma,t}i^\beta$ and $\Gamma_{\beta\gamma,t}di^\beta$

C.) the change in the Coriolis flux-density vector is $d(\bar{\psi}_{,t\alpha} - \bar{\psi}_{\gamma,t})$. In the equations occurs only $d\psi_{t,\alpha} = d(\Gamma_{\beta t,\alpha}i^\beta) = d\Gamma_{\beta t,\alpha}i^\beta + \Gamma_{\beta t,\alpha}di^\beta$.

To each of the six flux changes corresponds a change of voltage or torque. The three changes due to $dL_{\alpha\beta}$, $d\Gamma_{\beta\gamma,t}$ and $d\Gamma_{\beta t,\alpha}$ can be measured only by the moving observers. Hence for a machine with *stationary* coordinate axes the Equation of Small Oscillations reduces to

$$de_\alpha + \frac{\partial e_\alpha}{\partial x^\beta}dx^\beta = R_{\alpha\beta}di^\beta + L_{\alpha\beta}\frac{di^\beta}{dt} + \Gamma_{\beta\gamma,\alpha}di^\beta i^\gamma + \Gamma_{\beta\gamma,\alpha}i^\beta di^\gamma \quad ..145$$

A more detailed physical discussion is given in the previous papers.

XL. The Motional Impedance

a.) Let any machine with *stationary* coordinate axes and without sliprings be considered. If at the instant of disturbance the machine is in equilibrium, that is if all its currents are known, the Equation of Small Oscillation, equ. 103 can be written as

$$de_\alpha = (R_{\alpha\beta} + L_{\alpha\beta}p + \Gamma_{\beta\gamma,\alpha}i^\gamma + \Gamma_{\gamma\beta,\alpha}i^\gamma)di^\beta.$$

The expression in parenthesis is a tensor of rank two $\bar{Z}_{\alpha\beta}$ called the "motional impedance tensor" representing the opposition of a machine to a suddenly applied infinitesimal terminal voltage or shaft torque, allowing its speed also to vary, that is

$$\boxed{\bar{Z}_{\alpha\beta} = R_{\alpha\beta} + L_{\alpha\beta}p + \Gamma_{\beta\gamma,\alpha}i^\nu + \Gamma_{\gamma\beta,\alpha}i^\gamma} \quad ....146$$

Comparing it with the "transient impedance tensor" $Z_{\alpha\beta}$ equ. 64, it is found that $Z_{\alpha\beta}$ is only a special case of $\bar{Z}_{\alpha\beta}$, that is

$$\bar{Z}_{\alpha\beta} = Z_{\alpha\beta} + \Gamma_{\beta\gamma,t}i^\gamma + \Gamma_{\gamma t,\alpha}i^\gamma + \Gamma_{\gamma\beta,t}i^\gamma \quad ....147$$

$\bar{Z}_{\alpha\beta}$ differs from $Z_{\alpha\beta}$ only by an extra row and column corresponding to the additional axis t.

The Equation of Small Oscillations becomes

$$\boxed{de_\alpha = \bar{Z}_{\alpha\beta}di^\beta} \quad ....148$$

from which di^β is found as $de_\alpha\bar{Y}^{\alpha\beta}$ by calculating the inverse of $\bar{Z}_{\alpha\beta}$. For the representative machine

$$
\bar{Z}_{\alpha\beta} = \begin{array}{c|c|c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{d_s} & \multicolumn{1}{c}{d_r} & \multicolumn{1}{c}{q_r \quad {}^{\beta}} & \multicolumn{1}{c}{q_s} & \multicolumn{1}{c}{t} \\
\hline
d_s & r_{sd} + L_{sd}p & M_d p & -M_d p\theta & 0 & i^{qr} M_d \\
d_r & M_d p & r_r + L_{rd} p & -L_{rd} p\theta & 0 & i^{qr}(L_{rd} - L_{rq}) - i^{qs} M_q \\
q_r & 0 & L_{rq} p\theta & r_r + L_{rq} p & M_q p & i^{dr}(L_{rd} - L_{rq}) + i^{ds} M_d \\
q_s & 0 & M_q p\theta & M_q p & r_{sq} + L_{sq} p & -i^{dr} M_q \\
t & 0 & i^{qr} L_{rq} + i^{qs} M_q & -i^{dr} L_{rd} - i^{ds} M_d & 0 & r + Lp \\
\hline
\end{array} \quad \ldots 149
$$

For any machine or groups of machines with stationary coordinate axes $\bar{Z}_{\alpha\beta}$ is found by equ. 68.

b.) If the machine has sliprings or revolving brushes, but the coordinate axes are still assumed to be stationary, equ. 145 becomes

$$de_\alpha = \bar{Z}_{\alpha\beta}\, di^\beta - \frac{\partial e_\alpha}{\partial x^t}\, dx^t = \left(\bar{Z}_{\alpha\beta}\, p - \frac{\partial e_\alpha}{\partial x^\beta}\right) dx^\beta \dots\dots\dots 150$$

and the dyadic in parenthesis (not a tensor now) is the motional impedance. In actual calculations the form of the matrix may be simplified by assuming all currents $di^\beta(\beta = d_s, q_s, q_r, d_r)$ and dx^t as unknowns. For a salient pole synchronous machine this simplified form of motional impedance is (where $e_{dr} = e \sin(\theta_r - \theta) = e \sin \delta$, $e_{qr} = e \cos \delta$ hence $\partial e_{dr}/\partial x^t = -e \cos \delta$ and $\partial e_{qr}/\partial x^t = e \sin \delta$).

$$\bar{Z}_{\alpha\beta} = \begin{array}{c|c|c|c|c|c|}
 & d_s & d_r & q_r & q_s & t \\
\hline
d_s & r_{sd} + L_{sd}\,p & M_d\,p & -M_d\,p\theta & 0 & i^{qr} M_d \\
\hline
d_r & M_d\,p & r_r + L_{rd}\,p & -L_{rd}\,p\theta & 0 & i^{qr}(L_{rd} - L_{rq}) - i^{qs} M_q \\
\hline
q_r & 0 & L_{rq}\,p\theta & r_r + L_{rq}\,p & M_q\,p & i^{dr}(L_{rd} - L_{rq}) + i^{ds} M_d \\
\hline
q_s & 0 & M_q\,p\theta & M_q\,p & r_{sq} + L_{sq}\,p & -i^{dr} M_q \\
\hline
t & 0 & (i^{qr} L_{rq} + i^{qs} M_q)\,p + e\cos\delta & (-i^{dr} L_{rd} - i^{ds} M_d)\,p - e\sin\delta & 0 & (r + Lp)\,p \\
\hline
\end{array} \quad ..151$$

The currents, torque etc. calculated by this dynamical method check term by term with the expressions given by Park for the salient-pole alternator.

XLI. The Criterion of Dynamic Stability

If the coefficients in the motional impedance matrix of a machine are all real numbers (as in case of the salient-pole synchronous machine) an arithmetical method can be set up to investigate whether a machine will return to its former equilibrium position if it is disturbed by any cause whatever.

The determinant of the Motional Impedance $\bar{Z}_{\alpha\beta}$ has the form

$$D = a_0p^n + a_1p^{n-1} + \dots a_{n-1}p + a_n \qquad \text{152}$$

If all the real parts of the roots of the Determinantal Equation

$$|\bar{Z}_{\alpha\beta}| = 0 \qquad \text{153}$$

are negative, each current is of the form $A\epsilon^{(-\alpha \pm j\beta)t}$ all currents and speed variations will die out eventually and the machine is dynamically stable.

In order that the real parts of all roots should be negative, the coefficients of equ. 110 must satisfy the following conditions set up by Hurwitz.

1.) All the coefficients a_0, a_1, ... must be positive.

2.) All the following n determinants formed by the coefficients must be also positive

$$\Delta_\lambda = \begin{vmatrix} a_1a_3a_5 \dots a_{2\lambda-1} \\ a_0a_2a_4 \quad a_{2\lambda-2} \\ 0\ a_1a_3 \quad a_{2\lambda-3} \\ \vdots \\ 0 \dots\dots a_\lambda \end{vmatrix} \qquad \text{154}$$

where λ varies from 1 to n.

THE CURVATURE TENSOR

XLII. Variable Character of the Equations of Oscillations

a.) A very important property of the Equation of Small Oscillations (equs. 137 or 138) should be noted as opposed to that of the Equation of Motion:

1.) *No term of the equation is a tensor.*

2.) *No combination of terms can be set up that are tensors.*

From these facts it follows that in case of machines with *moving* coordinate axes each term must be transformed by a more or less complicated transformation formula. For instance $d\Gamma^{\alpha}_{\beta\gamma}i^{\beta}i^{\gamma}$ is to be transformed by

$$d\Gamma^{\delta}_{\epsilon\lambda}i^{\epsilon}i^{\lambda} = d\left(\Gamma^{\alpha}_{\beta\gamma}C^{\beta}_{\epsilon}C^{\gamma}_{\lambda}C^{\delta}_{\alpha} + \frac{\partial C^{\alpha}_{\epsilon}}{\partial x}C^{\delta}_{\alpha}\right)i^{\beta}C^{\epsilon}_{\beta}i^{\gamma}C^{\lambda}_{\gamma} = \Big[(d\Gamma^{\alpha}_{\beta\gamma})\,C^{\beta}_{\epsilon}C^{\gamma}_{\lambda}C^{\delta}_{\alpha}$$

$$+ \Gamma^{\alpha}_{\beta\gamma}dC^{\beta}_{\epsilon}C^{\gamma}_{\lambda}C^{\delta}_{\alpha} + \Gamma^{\alpha}_{\beta\gamma}C^{\beta}_{\epsilon}dC^{\gamma}_{\lambda}C^{\delta}_{\alpha} + \Gamma^{\alpha}_{\beta\gamma}C^{\beta}_{\epsilon}C^{\gamma}_{\lambda}dC^{\delta}_{\alpha} + \frac{\partial^2 C^{\alpha}_{\epsilon}}{\partial x^{\lambda}\partial x^{\sigma}}dx^{\sigma}C^{\delta}_{\alpha}$$

$$+ \frac{\partial C^{\alpha}_{\epsilon}}{\partial x^{\lambda}}dC^{\delta}_{\alpha}\Big]\,i^{\beta}C^{\epsilon}_{\beta}i^{\gamma}C^{\lambda}_{\gamma}\ldots\ldots\ldots\ldots\ldots\ldots\ldots 155$$

b.) However the Equation of Small Torques (equ. 143) is a *tensor* equation. The first three terms are evidently tensors and it will be proved now that the sum of the last three terms, representing a change in the machine torque, is also a tensor, that is both the stationary and the moving observers measure the same change of torque, which of course must be true from physical considerations also. By equations 42, 55 and 155, since C^{t}_{t} = unity

$$\Gamma_{\beta\gamma,t}di^{\beta}i^{\gamma} + \Gamma_{\beta\gamma,t}i^{\beta}di^{\gamma} + d\Gamma_{\beta\gamma,t}i^{\beta}i^{\gamma}$$

$$= \Gamma_{\epsilon\lambda,t}C^{\epsilon}_{\beta}C^{\lambda}_{\gamma}\,(di^{\sigma}C^{\beta}_{\sigma} + i^{\sigma}dC^{\beta}_{\sigma})\,i^{\pi}C^{\gamma}_{\pi} + \Gamma_{\epsilon\lambda,t}C^{\epsilon}_{\beta}C^{\lambda}_{\gamma}i^{\sigma}C^{\beta}_{\sigma}\,(di^{\pi}C^{\gamma}_{\pi} + i^{\pi}dC^{\gamma}_{\pi})$$

$$+ d\Gamma_{\epsilon\lambda,t}C^{\epsilon}_{\beta}C^{\lambda}_{\gamma}i^{\sigma}C^{\beta}_{\sigma}i^{\pi}C^{\gamma}_{\pi} + \Gamma_{\epsilon\lambda,t}dC^{\epsilon}_{\beta}C^{\lambda}_{\gamma}i^{\sigma}C^{\beta}_{\sigma}i^{\pi}C^{\gamma}_{\pi} + \Gamma_{\epsilon\lambda,t}C^{\epsilon}_{\beta}dC^{\lambda}_{\gamma}i^{\sigma}C^{\beta}_{\sigma}i^{\pi}C^{\gamma}_{\pi}$$

$$= (\Gamma_{\sigma\pi,t}di^{\sigma}i^{\pi} + \Gamma_{\sigma\pi,t}i^{\sigma}di^{\pi} + d\Gamma_{\sigma\pi,t}i^{\sigma}i^{\pi})$$

$$+ \Gamma_{\epsilon\lambda,t}C^{\lambda}_{\gamma}C^{\gamma}_{\pi}\,(C^{\epsilon}_{\beta}dC^{\beta}_{\sigma} + dC^{\epsilon}_{\beta}C^{\beta}_{\sigma}) + \Gamma_{\epsilon\lambda,t}C^{\epsilon}_{\beta}C^{\beta}_{\sigma}\,(C^{\lambda}_{\gamma}dC^{\gamma}_{\pi} + dC^{\lambda}_{\gamma}C^{\gamma}_{\pi}).$$

The last two expressions are zero since the terms in parenthesis are $d(C^{\epsilon}_{\beta}C^{\beta}_{\sigma}) = d(1) = 0$.

c.) *It is desirable to set up the Equation of Small Oscillations in a tensor or invariant form, that is in such a form that it should be possible to pass from one coordinate system to the other by the simple transformation formula of tensors* (equ. 36) *instead of complicated formulae* such as equ. 55.

Referring back to section XXXVIII and to equ. 140 the Equation of Small Oscillations was found from the Equation of Motion by taking the differentials of each term, that is by

$$de^{\alpha} = d\,(R^{\alpha}_{\cdot\beta}i^{\beta}) + d\left(\frac{\delta i^{\alpha}}{\delta t}\right).$$

This form suggests that *an invariant form of the Equation of Small Oscillations can be found by taking the "absolute" differential of each term instead of the ordinary differential.* That is

$$\delta e^\alpha = \delta\,(R^\alpha_{\cdot\beta} i^\beta) + \delta\left(\frac{\delta i^\alpha}{\delta t}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots 156$$

d.) It is necessary to have on the right hand side δi^α (or rather dx^α) as the independent variable. Hence it is necessary to change

1.) the tensor $\delta(\delta i^\alpha/\delta t)$ into the sum of two tensors so that one of the tensors should be $(\delta/\delta t)\delta i^\alpha$

2.) the tensor $\delta(R^\alpha_{\cdot\beta} i^\beta)$ into the sum of two tensors so that one of the tensors should be $R^\alpha_{\cdot\beta}\delta i^\beta$.

XLIII. The Generalized Riemann-Christoffel Curvature Tensor

a.) In order to find the difference between $\delta(\delta i^\alpha/\delta t)$ and $(\delta/\delta t)\delta i^\alpha$ let each be developed in its component parts.

$$\delta\left(\frac{\delta i^\alpha}{\delta t}\right) = d\left(\frac{\delta i^\alpha}{\delta t}\right) + \Gamma^\alpha_{\beta\gamma}\frac{\delta i^\beta}{\delta t}dx^\gamma = d\,\frac{di^\alpha}{dt} + d\Gamma^\alpha_{\beta\gamma} i^\beta i^\gamma + \Gamma^\alpha_{\beta\gamma} di^\beta i^\gamma$$

$$+ \Gamma^\alpha_{\beta\gamma} i^\beta di^\gamma + \Gamma^\alpha_{\beta\gamma}\frac{di^\beta}{dt}dx^\gamma + \Gamma^\alpha_{\beta\gamma}\Gamma^\beta_{\lambda\epsilon} i^\lambda i^\epsilon dx^\gamma$$

$$\frac{\delta}{\delta t}\delta i^\alpha = \frac{d}{dt}\delta i^\alpha + \Gamma^\alpha_{\beta\gamma}\delta i^\beta i^\gamma = \frac{d}{dt}di^\alpha + \frac{d\Gamma^\alpha_{\beta\gamma}}{dt} i^\beta dx^\gamma + \Gamma^\alpha_{\beta\gamma}\frac{di^\beta}{dt}dx^\gamma$$

$$+ \Gamma^\alpha_{\beta\gamma} i^\beta di^\gamma + \Gamma^\alpha_{\beta\gamma} di^\beta i^\gamma + \Gamma^\alpha_{\beta\gamma}\Gamma^\beta_{\epsilon\lambda} i^\epsilon dx^\lambda i^\gamma$$

$$\delta\left(\frac{\delta i^\alpha}{\delta t}\right) - \frac{\delta}{\delta t}\delta i^\alpha = \left(\frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\delta} + \Gamma^\alpha_{\lambda\gamma}\Gamma^\lambda_{\beta\delta} - \Gamma^\alpha_{\lambda\delta}\Gamma^\lambda_{\beta\gamma}\right) i^\beta i^\delta dx^\gamma.$$

The expression in parenthesis is a tensor of rank four, is denoted by $K_{\dot\delta\dot\gamma\beta}{}^\alpha$ *and called the "generalized Riemann-Christoffel curvature tensor" or shortly "curvature tensor." It plays a fundamental rôle in finding the conditions of integrability of a set of linear differential equations.* That is

$$\boxed{K_{\dot\delta\dot\gamma\beta}{}^\alpha = \frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\lambda\gamma}\Gamma^\lambda_{\beta\delta} - \Gamma^\alpha_{\lambda\delta}\Gamma^\lambda_{\beta\gamma}} \quad \ldots\ldots\ldots 157$$

Hence

$$\boxed{\delta\left(\frac{\delta i^\alpha}{\delta t}\right) - \frac{\delta(\delta i^\alpha)}{\delta t} = K_{\dot\delta\dot\gamma\beta}{}^\alpha i^\delta i^\beta dx^\gamma} \quad \ldots\ldots\ldots\ldots\ldots 158$$

b.) The difference between $\delta(R^{\alpha}_{\cdot\beta}i^{\beta})$ and $R^{\alpha}_{\cdot\beta}\delta i^{\beta}$ is $(\delta R^{\alpha}_{\ \beta})i^{\beta}$. But by equ. 123.

$$\delta(R^{\alpha}_{\cdot\beta})i^{\beta} = \delta(L^{\alpha\gamma}R_{\gamma\beta})i^{\beta} = (\delta L^{\alpha\gamma})R_{\gamma\beta}i^{\beta} + L^{\alpha\gamma}(\delta R_{\gamma\beta})i^{\beta}$$
$$= (dL^{\alpha\gamma} + L^{\sigma\gamma}\Gamma^{\alpha}_{\sigma\epsilon}dx^{\epsilon} + L^{\alpha\sigma}\Gamma^{\gamma}_{\sigma\epsilon}dx^{\epsilon})\,R_{\gamma\beta}i^{\beta} + L^{\alpha\gamma}\,(-\,\Gamma^{\sigma}_{\gamma\epsilon}R_{\sigma\beta} - \Gamma^{\sigma}_{\beta\epsilon}R_{\gamma\sigma})\,i^{\beta}dx^{\epsilon}$$
$$= \left(\Gamma^{\alpha}_{\lambda\gamma}R^{\lambda}_{\ \beta} - R^{\alpha}_{\ \lambda}\Gamma^{\lambda}_{\beta\gamma} + \frac{\partial L^{\alpha\lambda}}{\partial x^{\gamma}}R_{\lambda\beta}\right)i^{\beta}dx^{\gamma}.$$

The expression in parenthesis is a tensor of rank three, is denoted by $R_{\dot\gamma}{}_{\dot\beta}{}^{\alpha}$ and can be called the "resistance tensor of rank three" to differentiate it from the resistance tensor of rank two $R_{\alpha\beta}$. That is

$$\boxed{R_{\gamma}{}_{\dot\beta}{}^{\alpha} = \frac{\partial L^{\alpha\lambda}}{\partial x^{\gamma}}R_{\lambda\beta} + \Gamma^{\alpha}_{\lambda\gamma}R^{\lambda}_{\ \beta} - R^{\alpha}_{\ \lambda}\Gamma^{\lambda}_{\beta\gamma}} \quad \ldots\ldots\ldots\ldots 159$$

$$\boxed{\delta\,(R^{\alpha}_{\ \beta}i^{\beta}) - R^{\alpha}_{\ \beta}\delta i^{\beta} = R_{\gamma}{}_{\beta}{}^{\alpha}i^{\beta}dx^{\gamma}} \quad \ldots\ldots\ldots\ldots\ldots 160$$

XLIV. The Equation of Small Oscillations in Invariant Form

a.) If equations 158 and 160 are substituted into equation 156 the Equation of Small Oscillations becomes in terms of contravariant vectors

$$\boxed{\delta e^{\alpha} + \frac{\delta e^{\alpha}}{\delta x^{\beta}}dx^{\beta} = R^{\alpha}_{\ \beta}\delta i^{\beta} + \frac{\delta(\delta i^{\alpha})}{\delta t} + K_{\delta\gamma}{}_{\dot\beta}{}^{\alpha}i^{\delta}i^{\beta}dx^{\gamma} + R_{\dot\gamma}{}_{\dot\beta}{}^{\alpha}i^{\beta}dx^{\gamma}} \ .161$$

In terms of covariant vectors it is

$$\boxed{\delta e_{\alpha} + \frac{\delta e_{\alpha}}{\delta x^{\beta}}dx^{\beta} = R_{\alpha\beta}\delta i^{\beta} + L_{\alpha\beta}\frac{\delta(\delta i^{\alpha})}{\delta t} + K_{\delta\gamma\beta\alpha}i^{\delta}i^{\beta}dx^{\gamma} + R_{\gamma\beta\alpha}i^{\beta}dx^{\gamma}}$$

In the equations δi^{α} can not be replaced by $(\delta/\delta t)dx^{\alpha}$. The two expressions are not equal due to the asymmetry of $\Gamma^{\alpha}_{\beta\gamma}$.

In terms of ordinary differentials equ. 161 becomes (if the disturbing force δe^{α} is left out and only the free variation of e^{α} that is $(\delta e^{\alpha}/\delta x^{t})dx^{t}$ is considered)

$$\begin{aligned}
de^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}e^{\beta}dx^{\gamma} &= d(R^{\alpha}_{\ \beta}i^{\beta}) + \Gamma^{\alpha}_{\gamma\delta}R^{\gamma}_{\ \beta}i^{\beta}dx^{\delta} && \big\}\ \delta(R^{\alpha}_{\cdot\beta}i^{\beta}) \\
&+ \Gamma^{\alpha}_{\beta\gamma}e^{\beta}dx^{\gamma} - \Gamma^{\alpha}_{\gamma\delta}R^{\gamma}_{\cdot\beta}i^{\beta}dx^{\delta} + d(di^{\alpha})/dt + \Gamma^{\alpha}_{\beta\gamma}di^{\beta}i^{\gamma} + \Gamma^{\alpha}_{\beta\gamma}i^{\beta}di^{\gamma} && \Big\}\ \frac{\delta(\delta i^{\alpha})}{\delta t} \\
&+ (\partial\Gamma^{\alpha}_{\beta\gamma}/\partial x^{\delta} + \Gamma^{\alpha}_{\lambda\delta}\Gamma^{\lambda}_{\beta\gamma} - \Gamma^{\alpha}_{\lambda\gamma}\Gamma^{\lambda}_{\beta\delta})\,i^{\beta}i^{\delta}dx^{\gamma} && \\
&+ (\partial\Gamma^{\alpha}_{\beta\delta}/\partial x^{\gamma} - \partial\Gamma^{\alpha}_{\beta\gamma}/\partial x^{\delta} + \Gamma^{\alpha}_{\lambda\gamma}\Gamma^{\lambda}_{\beta\delta} - \Gamma^{\alpha}_{\lambda\delta}\Gamma^{\lambda}_{\beta\gamma})\,i^{\beta}i^{\delta}dx^{\gamma} && \big\}\ K_{\dot\delta\dot\gamma}{}_{\dot\beta}{}^{\alpha}i^{\delta}i^{\beta}dx^{\gamma} \ldots 162
\end{aligned}$$

It can be seen that the equation in invariant form consists of the previous equation 137 with additional terms *added* and the same terms *subtracted.* In equ. 137 the terms could not be grouped to form tensors but in the presence of additional terms representing hypothetical voltages and torques groups can be formed so that each group of terms is a tensor.

b.) In calculating all the possible coefficients $K_{\dot{\delta}\dot{\gamma}\beta}{}^{\alpha}$ the following should be noted

1.) $K_{\delta\dot{\gamma}\dot{\beta}}{}^{\alpha} = - K_{\dot{\gamma}\dot{\delta}\beta}{}^{\alpha}$. This is the only skew-symmetry that occurs in the generalized curvature tensor.

2.) α is always t. That is the curvature tensor is zero in the voltage equation.

3.) Either δ or γ is t, that is only $K_{\dot{t}\dot{\gamma}\beta}{}^{t}$ and $K_{\delta\dot{t}\dot{\beta}}{}^{t}$ exist where also $K_{\delta\dot{t}\dot{\beta}}{}^{t} = - K_{\dot{t}\dot{\gamma}\beta}{}^{t}$. Hence all possible coefficients can be represented in a matrix form $K_{\dot{t}\dot{\gamma}\beta}{}^{t}$ where only γ and β vary. All the other $(n^4 - n^2)$ coefficients are zero.

Hence the *Equation of Small Voltages* in invariant form is

$$\delta e^{\alpha} + \frac{\partial e^{\alpha}}{\partial x^{\beta}} dx^{\beta} = R^{\alpha}_{\cdot\beta}\delta i^{\beta} + \frac{\delta(\delta i^{\alpha})}{\delta t} + R_{\dot{\gamma}\dot{\beta}}{}^{\alpha} i^{\beta} dx^{\gamma} \quad \ldots\ldots\ldots 163$$

and the *Equation of Small Torques* in invariant form is

$$\delta e^{t} = R^{t}_{\cdot t}\delta i^{t} + \frac{\delta(\delta i^{t})}{\delta t} + K_{\delta\gamma\dot{\beta}}{}^{t} i^{\delta} i^{\beta} dx^{\gamma} + R_{\gamma\dot{\beta}}{}^{t} i^{\beta} dx^{\gamma} \quad \ldots\ldots\ldots 164$$

XLV. *Ricci Tensors*

a.) In general $\Gamma^{\alpha}_{\beta\gamma}$ is a function of all the variables and it can be differentiated with respect to *all* the variables. In the analysis of rotating electrical machinery however $\Gamma^{\alpha}_{\beta\gamma}$ can be differentiated only with respect to $x^{t} = \theta$, since it is a function only of θ. Due to this restricted variation, in calculating any coefficient in *any* coordinate system it is found that *two of the terms in equ. 157 is always zero,* that is

$$K_{\dot{t}\dot{\gamma}\dot{\beta}}{}^{t} = - \frac{\partial\Gamma^{t}_{\beta\gamma}}{\partial x^{t}} + \Gamma^{t}_{\lambda\gamma}\Gamma^{\lambda}_{\beta t} \quad \ldots\ldots\ldots\ldots\ldots 165$$

and

$$K_{\dot{\delta}\dot{t}\dot{\beta}}{}^{t} = \frac{\partial\Gamma^{t}_{\beta\delta}}{\partial x^{t}} - \Gamma^{t}_{\lambda\delta}\Gamma^{\lambda}_{\beta t} \quad \ldots\ldots\ldots\ldots\ldots 166$$

b.) *In going over from one coordinate system to the other by a non-holonomic transformation,* for instance from $K_{t\dot{m}\dot{n}}{}^{t}$ to $K_{\dot{t}\dot{\pi}\dot{\epsilon}}{}^{t}$, *the right* *hand side of the above equations does not behave as a tensor,* that is the transformation formula of $K_{t\,\sigma\,\pi}^{\cdot\,\cdot\,\cdot\,t}$ is

$$\Gamma^{t}_{\mu\sigma}\Gamma^{\mu\,\cdot}_{\pi\,t} - \frac{\partial\Gamma^{t}_{\pi\sigma}}{\partial x^{t}} = \left(\Gamma^{t}_{kn}\Gamma^{k}_{mt} - \frac{\partial\Gamma^{t}_{mn}}{\partial x^{t}}\right) C^{m}_{\pi}C^{n}_{\sigma} - \Gamma^{t}_{mn}\,\frac{\partial C^{n}_{\sigma}}{\partial x^{t}}\,C^{m}_{\pi} \quad \ldots 167$$

or

$$K_{t\,\sigma\,\pi}^{\cdot\,\cdot\,\cdot\,t} = K_{t\,n\,m}^{\cdot\,\cdot\,\cdot\,t} - \Gamma^{t}_{mn}\,\frac{\partial C^{n}_{\sigma}}{\partial x^{t}}\,C^{m}_{\pi} \quad \ldots\ldots\ldots\ldots\ldots 168$$

Similar relations hold true in the corresponding terms of $\delta(\delta i^{\alpha})/\delta t$.

Certain polyadics, among them the curvature tensor, that behave as tensors if the transformation is holonomic, loose their tensor character if the transformation is non-holonomic. Such tensors are called by Dienes "Ricci tensors." It can be seen that the tensors that appear in the analysis of infinitesimal displacements (hunting) like $K_{\delta\,\gamma\,\beta}^{\cdot\,\cdot\,\cdot\,\alpha}$ *and* $\delta(\delta i^{\alpha})/\delta t$ *are Ricci tensors.*

c.) *However their sum is always a true tensor,* which can be seen from the following considerations by reference to equ. 162.

1.) The last three terms of equ. 162 belonging to $K_{\delta\,\gamma\,\beta}^{\cdot\,\cdot\,\cdot\,\alpha}$ are cancelled always by a similar expression belonging to $\delta(\delta i^{\alpha})/\delta t$, that is the sum of the two sets of expression is always a so-called null-tensor with all its coefficients zero.

2.) The remaining expression $\partial\Gamma^{t}_{\beta\gamma}/\partial x^{t}$ belonging to $K_{\delta\,\gamma\,\beta}^{\cdot\,\cdot\,\cdot\,\alpha}$ always forms a tensor with the terms $\Gamma^{t}_{\beta\gamma}di^{\beta}i^{\gamma} + \Gamma^{t}_{\beta\gamma}i^{\beta}di^{\gamma}$ belonging to $\delta(\delta i^{\alpha})/\delta t$ as is proved in section XXXVIII b.

Hence if the values of $K_{\delta\,\gamma\,\beta}^{\cdot\,\cdot\,\cdot\,\alpha}i^{\delta}i^{\beta}dx^{\gamma}$ and $\delta(\delta i^{\alpha})/\delta t$ are calculated for any coordinate system, their value can be found for any other coordinate system as if each of them were tensors. This procedure is similar to that followed in the calculation of the Equation of Motion in finding separately di^{α}/dt and $\Gamma^{\alpha}_{\beta\gamma}i^{\beta}i^{\gamma}$ as if each of them were tensors (section XXXII d).

XLVI. Calculation of the Curvature Tensor

a.) For the representative machine with *moving* coordinate axes the coefficients of the curvature tensor as calculated from equ. 165 are

$$
K_{tmnt} = \begin{array}{c|c|c|c|c|}
{}_m \backslash {}^n & d_s & a & b & q_s \\
\hline
d_s & -M_d^2 L_{sq}/2B & -M_d L_{rd} L_{sq}\cos\theta/2B & M_d L_{rd} L_{sq}\sin\theta/2B & 0 \\
\hline
a & -M_d L_{rd} L_{sq}\cos\theta/2B & \begin{array}{c}\cos^2\theta(L_{rq}/2 - L_{rd}^2 L_{sq}/2B) \\ -\sin^2\theta(L_{rq}^2 L_{sd}/2A - L_{rd}/2)\end{array} & \begin{array}{c}\sin\theta\cos\theta(L_{rd}^2 L_{sq}/2B \\ -L_{rq}/2 - L_{rq}^2 L_{sd}/2A + L_{rd}/2)\end{array} & -M_q L_{rq} L_{sd}\sin\theta/2A \\
\hline
b & M_d L_{rd} L_{sq}\sin\theta/2B & \begin{array}{c}\sin\theta\cos\theta(L_{rd}^2 L_{sq}/2B \\ -L_{rq}/2 - L_{rq}^2 L_{sd}/2A + L_{rd}/2)\end{array} & \begin{array}{c}\cos^2\theta(L_{rd}/2 - L_{rq}^2 L_{sd}/2A) \\ -\sin^2\theta(L_{rd}^2 L_{sq}/2B - L_{rq}/2)\end{array} & -M_q L_{rq} L_{sd}\cos\theta/2A \\
\hline
q_s & 0 & -M_q L_{rq} L_{sd}\sin\theta/2A & -M_q L_{rq} L_{sd}\cos\theta/2A & -M_q^2 L_{sd}/2A \\
\hline
\end{array} \quad \ldots 169
$$

The coefficients of $K_{tmn}^{\cdot\cdot\cdot t}$ are the same as those of K_{tmnt} each divided by L.

b.) For the representative machine with *stationary* coordinate axes:

1.) if $K_{t\sigma\pi t}$ is calculated from equ. 169 assuming K_{tmnt} as a tensor

$$K_{t\sigma\pi t} = \begin{array}{c|c|c|c|c|} & d_s & d_r & q_r & q_s \\ \hline d_s & -M_d^2 L_{sq}/2B & -M_d L_{rd} L_{sq}/2B & 0 & 0 \\ \hline d_r & -M_d L_{rd} L_{sq}/2B & L_{rq}/2 - L_{rd}^2 L_{sq}/2B & 0 & 0 \\ \hline q_r & 0 & 0 & L_{rd}/2 - L_{rq}^2 L_{sd}/2A & -M_q L_{rq} L_{sd}/2A \\ \hline q_s & 0 & 0 & -M_q L_{rq} L_{sd}/2A & -M_q^2 L_{sd}/2A \\ \hline \end{array} \quad ..170$$

2.) if it is calculated from equ. 165

$$K_{t\sigma\pi t} = \begin{array}{c|c|c|c|c|} & d_s & d_r & q_r & q_s \\ \hline d_s & -M_d^2 L_{sq}/2B & -M_d L_{rd} L_{sq}/2B & 0 & 0 \\ \hline d_r & M_d/2 - M_d L_{rd} L_{sq}/2B & L_{rd}/2 - L_{rd}^2 L_{sq}/2B & 0 & 0 \\ \hline q_r & 0 & 0 & L_{rq}/2 - L_{rq}^2 L_{sd}/2A & M_q/2 - M_q L_{rq} L_{sd}/2A \\ \hline q_s & 0 & 0 & -M_q L_{rq} L_{sd}/2A & -M_q^2 L_{sd}/2A \\ \hline \end{array} \quad ..171$$

The difference between equs. 170 and 171 ought to be according to equ. 167

$$\Gamma^{t}_{mn} \frac{\partial C^{n}_{\sigma}}{\partial x^{t}} C^{m}_{\pi} = \begin{array}{c|c|c|c|c} & d_s & d_r & q_r & q_s \\ \hline d_r & -M_d/2 & (L_{rq} - L_{rd})/2 & 0 & 0 \\ \hline q_r & 0 & 0 & (L_{rd} - L_{rq})/2 & -M_q/2 \end{array} \quad \ldots 172$$

The sum of equs. 172 and 171 is actually equ. 170.

c.) The coefficients of $R_{\gamma\beta\alpha}$ are too lengthy to be published here.

DIFFERENTIAL GEOMETRY

XLVII. The Geometry of Holonomic Dynamics

a.) *In a Riemannian space there are two lines of great importance.* They are:

1.) *the geodesic line* (the tangent to the curve is always "absolutely" parallel to itself). Its equation is

$$\boxed{\frac{d^2x^\alpha}{ds^2} + \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0} \quad \ldots\ldots\ldots\ldots\ldots 173$$

2.) *a line at an infinitesimal distance from a geodesic line.* Its equation is

$$\boxed{\frac{\delta^2(dx^\alpha)}{\delta s^2} + K_{\delta\gamma\beta}^{\cdot\,\cdot\,\cdot\,\alpha} \frac{dx^\delta}{ds} \frac{dx^\beta}{ds} dx^\gamma = 0} \quad \ldots\ldots\ldots\ldots 174$$

b.) The set of differential equations of *holonomic* dynamics that are derived from the Equation of Motion of Lagrange can be considered as representing a particle describing a trajectory (path) under the action of a force f^α in an n-dimensional *Riemannian* space. The *equation of the trajectory* is

$$f^\alpha = \frac{d^2x^\alpha}{dt^2} + \begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix} \frac{dx^\beta}{dt} \frac{dx^\gamma}{dt} \ldots\ldots\ldots\ldots\ldots\ldots 175$$

The effect of a sudden infinitesimal disturbance is to force the particle into a path infinitely close to its previous path. Eventually this new path may diverge by a finite amount from its undisturbed path. The *equation of the disturbed path* (as given by Synge) is for a Riemannian space

$$\frac{\delta^2(dx^\alpha)}{\delta t^2} + K_{\delta\gamma\beta}^{\cdot\,\cdot\,\cdot\,\alpha} \frac{dx^\delta}{dt} \frac{dx^\beta}{dt} dx^\gamma - \frac{\delta e^\alpha}{\delta x^\beta} dx^\beta = 0 \ldots\ldots\ldots 176$$

This equation also can be written as

$$\frac{\delta}{\delta t}\,\delta\,\frac{dx^\alpha}{dt} + K_{\delta\gamma\beta}^{\cdot\cdot\cdot\alpha}\,\frac{dx^\delta}{dt}\,\frac{dx^\beta}{dt}\,dx^\gamma - \frac{\delta e^\alpha}{\delta x^\beta}\,dx^\beta = 0 \qquad \text{.........177}$$

since $\delta(dx^\alpha)$ is interchangeable with $d(\delta x^\alpha)$ in a Riemannian space.

c.) If it is assumed that the particle is subjected to a frictional force proportional to the velocities along the various axes, *the equation of the trajectory* is

$$\boxed{f^\alpha = R^\alpha{}_\beta\,\frac{dx^\beta}{dt} + \frac{d^2x^\alpha}{dt^2} + \begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}\frac{dx^\beta}{dt}\,\frac{dx^\gamma}{dt}} \qquad \text{...........178}$$

and *the equation of the disturbed path* is (replacing $\delta(R^{\cdot\alpha}_{\cdot\beta}i^\beta)$ by its two components)

$$\boxed{\frac{\delta}{\delta t}\,\delta\,\frac{dx^\alpha}{dt} + K_{\delta\gamma\beta}{}^\alpha\,\frac{dx^\delta}{dt}\,\frac{dx^\beta}{dt}\,dx^\gamma + R_{\gamma\beta}{}^\alpha\,\frac{dx^\beta}{dt}\,dx^\gamma + R^{\cdot\alpha}_{\cdot\beta}\,\frac{dx^\beta}{dt} - \frac{\delta e^\alpha}{\delta x^\beta}\,dx^\beta = 0} \;.\,179$$

d.) These equations are identical in form with the Equation of Motion and the Equation of Small Oscillations developed in the paper except $\{{}^{\alpha}_{\beta\gamma}\}$ is replaced by $\Gamma^\alpha_{\beta\gamma}$. Since in the equations of the machines $\Gamma^\alpha_{\beta\gamma}$ is asymmetrical, the analogous dynamical problem can be stated as follows:

The equations of rotating electrical machinery during accelerations are identical with the equations of a particle moving in an n-dimensional non-Riemannian space with asymmetric connection acted upon by a positional force and opposed by a frictional force proportional to its instantaneous velocity.

XLVIII. The Geometry of Rotating Electrical Machinery

a.) The line-element of the surface on which the particle moves is *defined* as

$$ds^2 = L_{\alpha\beta}dx^\alpha dx^\beta \qquad \text{.....................180}$$

As long as the metric tensor of the surface $L_{\alpha\beta}$ and the coefficients of connection $\Gamma_{\alpha\beta,\gamma}$ are defined in any arbitrary manner, *it is not necessary to know the relations between the variables x^α.* All (metric) properties of the surface and of vectors on the surface in the neighborhood of a point can be studied with the aid of the two sets of quantities $L_{\alpha\beta}$ and $\Gamma_{\alpha\beta,\gamma}$ one having n^2 the other n^3 terms.

b.) The projection of the instantaneous *position* of the particle along the curvilinear coordinate axes represent the total charges that passed through each winding.

The unit tangent vector to the trajectory is

$$\lambda^\alpha = \frac{dx^\alpha}{ds} \qquad \text{181}$$

c.) The *velocity* vector is along the tangent of the trajectory, that is

$$v^\alpha = i^\alpha = \frac{dx^\alpha}{dt} = \frac{dx^\alpha}{ds}\frac{ds}{dt} = \lambda^\alpha v, \qquad \text{182}$$

The projection of the velocity vector along the coordinate axes represents the instantaneous currents flowing in each winding. The magnitude of the velocity vector is

$$v^2 = \frac{ds^2}{dt^2} = L_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = L_{\alpha\beta}i^\alpha i^\beta = 2T \qquad \text{183}$$

That is the square of the velocity vector is equal to twice the kinetic energy stored in the machine at that instant.

d.) The *acceleration* vector e^α can be divided into $R^\alpha_{\cdot\beta} i^\beta$ which is in a general direction and into $\delta i^\alpha/\delta t$

$$\frac{\delta i^\alpha}{\delta t} = \frac{d(\lambda^\alpha v)}{dt} + \Gamma^\alpha_{\beta\gamma}\lambda^\beta\lambda^\gamma v^2 = \left(\frac{dv}{dt}\lambda^\alpha + v^2\frac{d\lambda^\alpha}{ds}\right) + v^2\Gamma^\alpha_{\beta\gamma}\lambda^\beta\lambda^\gamma = \frac{dv}{dt}\lambda^\alpha + v^2 h\mu^\alpha$$

where

$$h\mu^\alpha = \frac{d\lambda^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma}\lambda^\beta\lambda^\gamma = \frac{\delta\lambda^\alpha}{\delta s} \qquad \text{184}$$

$h\mu^\alpha$ is the "first curvature vector," μ^α is the "principal normal" and h is the "first curvature." Hence the voltage vector $\delta i^\alpha/\delta t$ (acceleration vector) due to the presence of flux lines lies in the plane of λ^α and μ^α.

e.) The *power* is represented by the product of the acceleration vectors and their projection along the velocity vector λ^α. Since the power represented by $T_{\alpha\beta\gamma}i^\alpha i^\beta i^\gamma$ is zero (equ. 107) the generated voltage vector due to the motion of rotor conductors and the torque vector must lie in a plane perpendicular to the direction of motion that is to λ^α.

Hence the product of $\delta i^\alpha/\delta t$ and its projection on λ^α gives dT/dt while the product of e^α and its projection on λ^α gives the power input.

f.) It is interesting to note that all physical space vectors actually existing inside of all machines at each instant (current-density, flux-density, voltage) are represented geometrically by vectors located in an n-dimensional space or upon an m-dimensional surface in an n-dimensional space (velocity, acceleration). The space or surface itself is represented by the charges that do not form a space wave inside the machine. Also all the properties of the space itself and of the machine can be investigated if the metric and the coefficients of connection are given, the particle and the machine are at rest and no vectors appear in the space or inside the machine. *Vectors do appear in the space and inside the machine only when a voltage is applied to the machine and the particle begins to move under the action of a force,* the vectors showing the instantaneous velocity, and acceleration of the particle. The manner of the motion of the particle of course depends on the type of the applied forces but the characteristic properties of the space itself in which the motion is described are independent of the applied forces, they only depend on the metric and connection.

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Guidelines to Antigravity*

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This paper emphasizes certain little known aspects of Einstein's general theory of relativity. Although these features are of minor theoretical importance, their understanding and use can lead to the generation and control of gravitational forces. Three distinctly different non-Newtonian gravitational forces are described. The research areas which might lead to methods for the control of gravitation are pointed out and guidelines for initial investigation into these areas are given.

INTRODUCTION

EINSTEIN'S general theory of relativity provides a number of ways to generate non-Newtonian gravitational forces. Theoretically, all of these forces could be used to counteract the gravitational field of the earth, thus acting as a form of antigravity. The three outlined here were probably known by Einstein before he published his paper on the principle of general relativity in 1916.[1] They were first specifically derived by Thirring[2] in 1918, and since then have been contained in nearly every text on general relativity.[3–5]

Although non-Newtonian forces are well known to the theorists in general relativity, they are little known to those outside the field, and it was for this reason that it was felt that a simplified discussion of these unusual gravitational effects would be of interest to the nonspecialist.

The equations of general relativity not only predict the usual radial Newtonian gravitational force behavior of a stationary mass on a stationary test body, but they also predict that a moving mass can create forces on a test body which are similar to the usual centrifugal and Coriolis forces, although much smaller. In addition, when the general relativity field equations are linearized, they result in a set of dynamic gravitational field relations similar to the Maxwell relations. Thus one can use intuitive pictures from electromagnetic theory to design theoretical models. Whether the effects predicted by the linearized theory really exist, will, of course, have to be checked by repeating the calculations with the nonlinearized field equations.

The essential point is that all of these unusual forces create accelerations which are independent of the mass of the test body and the forces are, therefore, indistinguishable from the usual Newtonian gravitational force.

NON-NEWTONIAN GRAVITATIONAL FORCES

Effect of Rotating Masses on Stationary Bodies

By using Einstein's general theory of relativity for a rotating system of masses, it can be shown that in addition to the usual Newtonian term, the gravitational scalar potential contains terms which arise from the rotation of the body. One of the shapes which has been rigorously investigated is the rotating massive ring.[3]

For a massive ring rotating in the x-y plane, the non-Newtonian acceleration on a stationary test body near the origin is approximately

$$\ddot{x} = (MG\omega^2/2c^2R)x,$$
$$\ddot{y} = (MG\omega^2/2c^2R)y, \tag{1}$$

and

$$\ddot{z} = -(MG\omega^2/c^2R)z.$$

Where M and R are the mass and radius of the ring, ω is the angular velocity, $G=6.67\times10^{-11}$ m^3/kg-sec^2 is the Newtonian gravitational constant, c is the speed of light, and x, y, z are the coordinates of the test body with respect to the origin of the rotating mass. From these equations it is evident that the rotating mass not only

* This paper was submitted as an essay to the Gravity Research Foundation in 1962 and received the Second Award.

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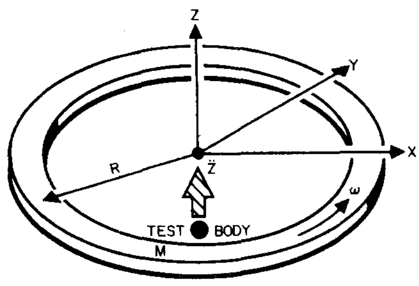

FIG. 1. Non-Newtonian component of acceleration on a test body below the center of a rotating massive ring.

forces the test body away from the axis in an imitation of centrifugal force, but also pulls it upward into the plane of rotation as shown in Fig. 1.

Effect of Rotating Masses on Moving Bodies

It has been pointed out that a rotating mass will exert a force similar to centrifugal force on a stationary test body; also, if the test body is moving at some constant velocity **v**, it will experience an additional force which is proportional to the cross product of the angular velocity of the rotating mass and the linear velocity of the test body. The additional force can be compared with two others: mechanically, it acts like a very weak Coriolis force; electrically,[4,6] it acts like the gravitational equivalent of the Lorentz force on a charged particle moving through a magnetic field.

One of the shapes which has been investigated is the rotating massive spherical shell. The acceleration on a test body moving with a velocity **v** inside the shell is approximately[4]

$$\ddot{x} = (GM/3c^2R)[\tfrac{4}{5}\omega^2 x - 8\omega v_y],$$
$$\ddot{y} = (GM/3c^2R)[\tfrac{4}{5}\omega^2 y + 8\omega v_x], \quad (2)$$
and
$$\ddot{z} = -(8GM\omega^2 z/15c^3R),$$

where v_x and v_y are the x and y components of the velocity of the test body and M and R are the mass and radius of the spherical shell. The first term in each expression is the centrifugal-type force on a stationary test body described in the previous section. The second term in the x and y components of the acceleration depends upon the velocity of the test body as shown in Fig. 2.

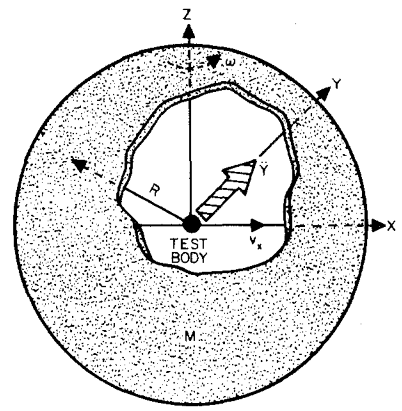

FIG. 2. Non-Newtonian component of acceleration on a test body with a velocity v_x at the center of a rotating massive sphere.

Effect of Accelerated Masses on Stationary Bodies

In employing Einstein's Theory to investigate the effect of a large accelerated mass on a small test body, it is found that the accelerated body drags the test body along with it. The exact equations are[4]

$$\ddot{\mathbf{r}} = \nabla(GM/R) + (4GM/c^2R)\mathbf{a} + (4GM/c^3)(\mathbf{a}\cdot\mathbf{R}/R^2)\mathbf{v}, \quad (3)$$
$$= F_1/m + F_2/m + F_3/m,$$

where M, **a**, and **v** are the mass, acceleration, and velocity of the large body, respectively, and **R** is the distance from the small body to the large body. In addition to the usual Newtonian attraction, the test body experiences forces in the direction of the acceleration and the velocity of the large body as shown in Fig. 3.

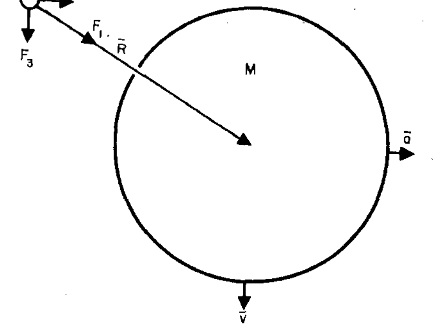

FIG. 3. Forces on a test body near a massive, accelerating, moving body.

[6] Robert L. Forward, Proc. IRE **49**, 892 (1961).

CREATING NON-NEWTONIAN GRAVITATIONAL FORCES

Devices Using Moving Masses

The equations in the previous section contain two common factors. One is the Newtonian gravitational field,

$$GM/r^2 \approx (4\pi/3)G\rho r, \tag{4}$$

and the other is the ratio of the characteristic system velocity to the velocity of light such as

$$v^2/c^2 \quad \text{or} \quad ar/c^2. \tag{5}$$

In order to obtain measurable amounts of gravitational force, these quantities must be as large as possible. To obtain a high gravitational field, either a large mass or a high density is required. The greater the density, the less total mass necessary to achieve the same gravitational field.

To obtain high rotational velocities, we cannot use the mechanical strength of materials since this limits the obtainable equatorial velocities to approximately the speed of sound.[7] Because of this, it will be necessary to use fields to hold the systems together under inertial stresses. One example would be the high gravitational fields obtainable with dense matter. However, any practical device which might be constructed would probably use electric or magnetic fields. By using electromagnetic forces to contain rotating systems, it would be possible for the masses to reach relativistic velocities; thus a comparatively small amount of matter, if dense enough and moving fast enough, could produce usable gravitational effects.

An example of a system held together by gravitational fields is a contact binary dwarf star system. The force equation describing the mutual rotation of the stars is

$$-(GMM/(2r)^2) = -M\omega^2 r = -(Mv^2/r). \tag{6}$$

A particle coming near the system will experience not only a radial acceleration[4]

$$\ddot{x}_r = -GM\left[\frac{1}{(b-r)^2}+\frac{1}{(b+r)^2}\right] - \frac{4GMa}{c^2}\left[\frac{1}{b-r}-\frac{1}{b+r}\right], \tag{7}$$

but also a tangential acceleration

$$\ddot{x}_t = \frac{4GMva}{c^3}\left[\frac{1}{b-r}+\frac{1}{b+r}\right] \approx \frac{1}{c^3 b}\left(\frac{GM}{r}\right)^{\frac{3}{2}}, \tag{8}$$

where b is the distance of the particle from the center of the system.

In applying these equations to a test object (such as a space vehicle) passing by this star system, it is evident that, in general, the radial acceleration will not introduce any net change in velocity, but that the tangential acceleration will transfer energy and momentum to the vehicle. For a neutron binary star, this acceleration can be greater than one million g's; although these accelerations seem very high, there are no stresses on the human body because the forces are gravitational. Such systems could be used to accelerate space vehicles to nearly the speed of light.

Devices Based on Analogies between Electromagnetic and Gravitational Fields

There are other types of devices for obtaining non-Newtonian gravitational forces which follow from the known electromagnetic analogies based on Einstein's general theory of relativity; however, these devices have not been analyzed using the complete field equations. For instance, two rotating gyroscopes should repel each other if oriented properly, and two pipes with massive liquid flowing through them should exhibit a pinch effect.

Although the rotating gyroscopes and flowing masses should exhibit non-Newtonian forces, the linearized theory does not predict an interaction with a stationary test body. A device which electromagnetic analogies predict might be able to create a non-Newtonian gravitational force that will accelerate a nonrotating, nonmoving body is one that contains accelerated masses whose mass flow is like the current flow in a wire wound torus.[8] If we look at the electromagnetic model, a current **I** through the wire causes a magnetic field in the torus. If the current constantly increases, then the magnetic field also increases with time. This time-varying magnetic field then creates a dipole electric field. (Fig. 4.)

[7] J. W. Beams, Sci. Am. **204**, No. 4, 134 (1961).

[8] Robert L. Forward, Proc. IRE **44**, 1402 (1961).

The value of the electric field at the center of the torus is

$$E = -\dot{B} = +(d/dt)(\mu N I r^2/4\pi R^2), \tag{9}$$

where R is radius of the torus, r is the radius of one of the loops of wire wound around it, and N is the total number of turns.

If we replace the wires with pipes carrying a massive liquid, then the known analogy between the electromagnetic and gravitational fields can now be used.[6] All the electromagnetic quantities are transformed to their equivalent gravitational quantities to obtain

$$G = -\dot{K} = -\frac{d}{dt}\left(\frac{\eta N T r^2}{4\pi R^2}\right), \tag{10}$$

where G is the gravitational field generated by the total mass current NT, and $\eta = 3.73\times10^{-26}$ m/kg is the gravitational equivalent to the magnetic permeability.[6] (See Fig. 5.)

It is important to notice in the above equation that the time derivative operates on the entire quantity in the brackets. One would normally say that all quantities are independent of time except the mass flow T. If the amount of mass flow increases with time, then $\dot{T}>0$, and thus a gravitational field G can be generated. However, in electromagnetism, the permeability of some materials such as iron is nonlinear, which allows the construction of highly efficient electromagnetic field generators. A material with a highly nonlinear η would also be useful in the construction of efficient gravitational field generators.

RESEARCH AREAS

Dense Materials

It has been emphasized that in order to achieve measurable gravitational effects with moderate amounts of mass, dense matter is required. Thus, the study of degenerate matter could lead to the generation and control of gravitation. Methods must be found to manufacture, contain, and control matter with densities from 10^8 to 10^{15} g/cm^3. The best starting point appears to be an investigation of the neutron–neutron interaction. By using extremely low temperatures and strong magnetic fields, one should be able to cool the thermal neutrons from a pile and concentrate them into a small region through the interaction of the magnetic field with the magnetic moment of the neutron.[9] The Fermi energy will limit the density to about 10^{-6} g/cm^3, but the formation of tetraneutrons (n^4) or the existence of a superconductive-type phase space condensation will create bosons which do not have this limitation. The results of a more comprehensive study of the properties of a cold neutron gas and the methods for containment will be given in a future paper.

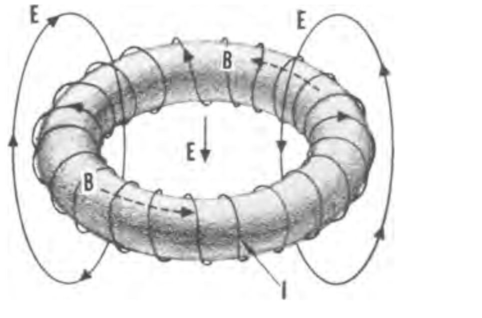

FIG. 4. Generator of a dipole electric field.

Gravitational Properties of Matter

In studying analogies between electromagnetism and gravitation, it can be seen that one analogous quantity has not been investigated. This is the gravitational equivalent to the magnetic permeability. Electrical power distribution systems depend upon the anomalously large and nonlinear permeability of iron and other magnetic materials. Since all atoms have spin, all materials will have a gravitational permeability which is different from that of free space. Rough calculations show that this difference is very small, but experimental investigation may

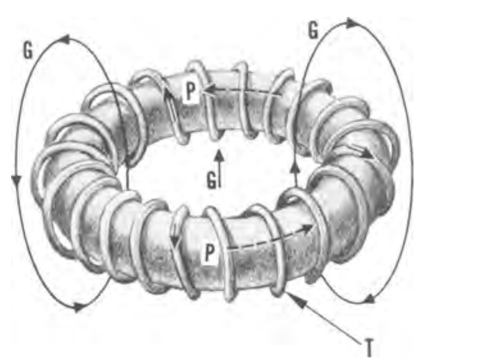

FIG. 5. Generator of a dipole gravitational field.

[9] V. V. Vladimirskii, Soviet Phys.—JETP **12**, 740 (1961).

find materials with anomalously large or nonlinear properties that can be used to enhance time-varying gravitational fields. Also, since the magnetic moment and the inertial moment are combined in an atom, it may be possible to use this property to convert time-varying electromagnetic fields into time-varying gravitational fields. At present, the only way to search for such materials is to intersperse wedges of material between gravitational wave generators and detectors, such as those described by J. Weber,[10] and look for a change in amplitude or direction of propagation. The first efforts in this direction[11] have been carried out by the Russian workers Braginsky, Rudenko, and Rukman with negative results.

It is obvious that research in the field of gravitation will be very difficult since even the most optimistic calculations indicate that very large devices will be required to create usable gravitational forces. Antigravity, like space travel, will probably have no direct effect on the daily life of the average person. Future progress in the control of gravitation, like all modern sciences, will require special projects involving large sums of money, men, and energy.

[10] J. Weber, Phys. Rev. **117**, 306 (1960).
[11] V. B. Braginsky, V. N. Rudenko, and G. I. Rukman, Zhur. Eksp. i Theoret. Fiz. **43**, 51 (1962) (in Russian with English abstract).

Lagrangian Solutions of the Equations of Motion in Fluid Mechanics by Integrating Certain Types of Nonlinear Ordinary Differential Equations

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(Received 16 July 1962)

The classical problems of the forced motion of a fluid particle induced by the motion of solid bodies immersed in an incompressible and nonviscous fluid of unbounded extent are solved by integrating nonlinear equations of the type:

$$(d^2r/dt^2)+P(r,t)(dr/dt)^3+Q(r,t)(dr/dt)^2+R(t)dr/dt=0,$$

and

$$(d^2r/dt^2)+M(r)(dr/dt)^2+N(r,t)dr/dt+S(r,t)=0.$$

The solutions of the Lagrangian equations of motion instead of the usual Euler's equations furnish better physical insight. The second integrals are evaluated in terms of both elliptic integrals and series for the problem of cylinder, and hyperelliptic integrals and series for the sphere. A systematic method of integrating nonlinear equations of this type is discussed. The problem of a half-body is also discussed.

I. INTRODUCTION

We propose to investigate first a class of equations which are nonlinear in one coordinate frame formed by dependent and independent variables and integrated by choosing a new coordinate frame formed by new dependent and independent variables. That is to say, the nonintegrability of the original equation is due to an improperly chosen coordinate frame. We wish to study systematically the method of finding the right frame.

In this paper, we consider a class of equations having the forms:

$$(d^2r/dt^2)+P(r,t)(dr/dt)^3+Q(r,t)(dr/dt)^2+R(t)(dr/dt)=0, \quad \text{(I)}$$

$$(d^2r/dt^2)+M(r)(dr/dt)^2+N(r,t)dr/dt+S(r,t)=0. \quad \text{(II)}$$

Equations (I) and (II) appear often in applied mechanics. We apply this method to the problem of the forced motion of fluid particles of a non-

General Relativity and Gravitation, Vol. 16, No. 8, 1984

On the Gravitational Effects of Rotating Masses: The Thirring–Lense Papers[1]

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Received May 30, 1982

Abstract

The purpose of this work is to provide a critical analysis of the classical papers of H. Thirring [*Phys. Z.*, **19**, 33 (1918); *Phys. Z.*, **22**, 29 (1921)] and J. Lense and H. Thirring [*Phys. Z.*, **19**, 156 (1918)] on rotating masses in the relativistic theory of gravitation and to render them accessible to a wider circle of scholars. An English translation of these papers is presented which follows the original German text as closely as possible. This is followed by a concise account of the significance of the results of these papers as well as the possibility of measuring the gravitational effects of rotating masses.

§(1): *Introduction*

Hans Thirring (1888–1976) was born and educated in Vienna. He studied mathematics and physics at the University of Vienna and received his Ph.D. in 1911. His habilitation thesis of 1915 was on the theory of the specific heat of solids. His major fields of interest were the theory of relativity and applied physics; moreover, he had wide-ranging interests in psychology, politics, and in issues related to world energy resources. He became professor of physics in Vienna in 1921. For political reasons he had to leave the University in 1938. In 1945 he was reinstated and became the director of the Institute for Theoretical Physics. Among the books he authored there is one entitled *Die Idee der Relativitätstheorie*.

Josef Lense (b. 1890) was born and educated in Vienna. He studied astronomy at the University, obtaining his Ph.D. in 1914 [cf. Die jovizentrische Bewe-

[1] Dedicated to Professor Josef Lense on the occasion of his 92nd birthday.

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gung der kleinen Planeten, *Astron. Nachrichten*, **196**, 341-346 (1913)]. His habilitation thesis of 1921 on the integration of differential expressions, as well as his subsequent work on quadratic forms, was motivated by the geometric aspects of the relativity theory. He became a lecturer in Vienna in 1921, and a professor of mathematics at the Technical University of Munich in 1927. In addition, from 1946 until his retirement, he was the director of the Mathematical Institute in Munich. He has authored many books in mathematics, a subject in which his major fields of interest have been differential geometry and mathematical physics. He presently makes his home in Munich.

The collaboration of Lense and Thirring began in Vienna after Thirring, aware of Lense's background in astronomy, inquired whether Lense would be willing to carry through, in perturbation theory, the integration of the equations of motion of a test body in the field of a rotating mass together with the application of the results to the orbits of the planets and moons [1]. This resulted in a joint publication in the *Physikalische Zeitschrift* of 1918.

§(2): *English Translation of the Thirring-Lense Papers*

On the Effect of Rotating Distant Masses in Einstein's Theory of Gravitation

HANS THIRRING

The considerations of this study can best be made clear by a quotation from Einstein's fundamental paper of 1914.[1] He says the following in the introduction:

"At first it seems that such an extension of the theory of relativity has to be rejected for physical reasons. Namely, let K be a permissible coordinate system in the Galilei-Newtonian sense, K' a uniformly rotating coordinate system with respect to K. Then centrifugal forces act on masses which are at rest relative to K', while not on masses at rest relative to K. Already Newton considered this as proof that the rotation of K' has to be interpreted as "absolute," and that one cannot thus treat K' as "at rest" with the same justification as K. This argument is not valid, however, as explained by E. Mach in particular. That is, we need not attribute the existence of those centrifugal forces to the motion of K'; rather, we can attribute them as well to the average rotational motion of the ponderable distant masses in the surrounding relative to K', whereby we treat K' as at rest. If the Newtonian laws of mechanics do not admit such a conception, this could well be caused by the defects of this theory. . . ."

Since Einstein's theory seems to have been brought to completion in the

[1] A. Einstein, *Berl. Ber.*, 1914, p. 1030; see also *Ann. Phys. (Leipzig)*, **49**, p. 769 (1916).

publications of 1915, the question suggests itself: Is the new theory devoid of the defects of the Newtonian theory insofar as the rotation of distant masses, according to their equations, indeed produces a gravitational field equivalent to a "centrifugal field"? Perhaps one could regard a discussion of this question futile by stating that the required equivalence is guaranteed by the general covariance of the field equations. But things are not quite as simple since the boundary conditions for $g_{\mu\nu}$ at spatial infinity play a role as well. These questions of principal interest have been treated in the papers of De Sitter[1] and Einstein.[2] We will not go into these general questions, rather we will calculate and study the field of rotating distant masses using a specific and concrete example. For this purpose, the method of approximate integration of the field equations given by Einstein[3] is perfectly appropriate. This method will serve as the foundation for the following calculations. We choose as an example the field inside a uniformly rotating infinitely thin hollow sphere endowed with a constant mass density.

In the first section of this paper (which can be skipped without affecting the understanding of the rest) the $g_{\mu\nu}$ inside the spherical shell will be calculated approximately; in the second section the motion of a point mass in this field will be discussed.

A. Calculational Part: Calculation of the $g_{\mu\nu}$ for the Neighborhood of the Center of the Rotating Hollow Sphere

Notation:

a, radius of the hollow sphere;
M, its mass;
ω, its angular velocity;
x, y, z, rectangular coordinates of a point on the surface of the sphere;
x_0, y_0, z_0, coordinates of the point under consideration;
χ, gravitational constant;
ρ_0, naturally measured space density of matter.

In connection with the approximations used for calculating the field, the following should be mentioned in advance: The field in the neighborhood of the center of the sphere will be considered to be so weak that in the field equations only terms of first order with respect to $\gamma_{\mu\nu}$ are taken into account ($\gamma_{\mu\nu}$ is defined by $g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}$). Since the higher-order terms are neglected, it is possible to use Einstein's method for the approximate integration of the field

[1] W. de Sitter, *Amsterdam Proc.*, **19**, 527, 1917.
[2] A. Einstein, *Berl. Ber.*, 1917, p. 142.
[3] A. Einstein, *Berl. Ber.*, 1916, p. 688.

equations. According to the second type of approximation often used, the components of the velocity of the ponderable matter will be considered as small with respect to unity (velocity of light), so that in this crudest approximation, which yields Newton's theory, already the first powers can be neglected. Let us apply this approximation–which is totally independent of the first one–insofar as we drop the terms of third and higher orders in the velocities in comparison to 1. Finally, our calculations relate to the neighborhood of the center of the sphere; let r be the distance between the point under consideration and the center of the sphere and let R be the distance between the point under consideration and the integration element. Then we will expand $1/R$ in a power series in r/a up to second order.

Einstein's method of approximate integration yields for the computation of the $g_{\mu\nu}$ the following formulas:

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}, \qquad \delta_{\mu\nu} = 1, \quad \mu = \nu \tag{1}$$

$$= 0, \quad \mu \neq \nu$$

$$\gamma_{\mu\nu} = \gamma_{\mu\nu}' - \frac{1}{2}\sum_{\alpha} \gamma_{\alpha\alpha}' \tag{2}$$

$$\gamma_{\mu\nu}' = -\frac{\chi}{2\pi}\int \frac{T_{\mu\nu}(x, y, z, t - r)}{R}\, dV_0 \tag{3}$$

Here $T_{\mu\nu}$ denotes the covariant energy tensor of matter and dV_0 the spatial volume element of integration space,

$$R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

The coefficients $g_{\mu\nu}$ of the line element refer to the coordinates

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = it$$

According to the first type of approximation, we are allowed to substitute the covariant energy tensor by the contravariant one. Neglecting the stresses, it is given by

$$T_{\mu\nu} = T^{\mu\nu} = \rho_0 \frac{dx_\mu}{ds}\frac{dx_\nu}{ds}$$

$$= \rho_0 \frac{dx_\mu}{dx_4}\frac{dx_\nu}{dx_4}\left(\frac{dx_4}{ds}\right)^2 \tag{4}$$

Let the hollow sphere rotate with angular velocity ω about the z axis, then for one of its points with polar coordinates, a, ϑ, φ, we have

$$\frac{dx_1}{dx_4} = -i\frac{dx}{dt} = ia\omega \sin\vartheta \sin\varphi$$

$$\frac{dx_2}{dx_4} = -i\frac{dy}{dt} = -ia\omega \sin\vartheta \cos\varphi \tag{5}$$

$$\frac{dx_3}{dx_4} = 0$$

These values upon substitution into (4) yields the following scheme for $T_{\mu\nu}$:

$$T_{\mu\nu} = \rho_0 \left(\frac{dx_4}{ds}\right)^2$$

$$\cdot \begin{Bmatrix} -a^2\omega^2 \sin^2\vartheta \sin^2\varphi & +a^2\omega^2 \sin^2\vartheta \sin\varphi\cos\varphi & 0 & ia\omega\sin\vartheta\sin\varphi \\ +a^2\omega^2 \sin^2\vartheta \sin\varphi\cos\varphi & -a^2\omega^2\sin^2\vartheta\cos^2\varphi & 0 & -ia\omega\sin\vartheta\cos\varphi \\ 0 & 0 & 0 & 0 \\ ia\omega\sin\vartheta\sin\varphi & -ia\omega\sin\vartheta\cos\varphi & 0 & 1 \end{Bmatrix} \tag{6}$$

Since we take ρ_0 to be the naturally measured matter density, one has to insert the naturally measured spatial volume element for dV_0 as well in order to ensure the tensorial character of the integral (3). For this the following formula[1] holds:

$$dV_0 = \sqrt{g}\, i\frac{dx_4}{ds} dV \tag{7}$$

For the integration we introduce polar coordinates. Thus

$$\sqrt{g}\, dV = a^2 da \sin\vartheta\, d\vartheta\, d\varphi \tag{8}$$

Finally, one still has to express $1/R$ in terms of the integration variables. We choose the coordinate system such that the point under consideration is situated in the Z-X plane. Its coordinates are then

$$x_0 = r\sin\vartheta_0, \quad y_0 = 0, \quad z_0 = r\cos\vartheta_0$$

Then we have

$$\begin{aligned} R^2 &= (a\sin\vartheta\cos\vartheta - r\sin\vartheta_0)^2 + a^2\sin^2\vartheta\sin^2\varphi + (a\cos\vartheta - r\cos\vartheta_0)^2 \\ &= a^2\left[1 - \frac{2r}{a}(\sin\vartheta\cos\varphi\sin\vartheta_0 + \cos\vartheta\cos\vartheta_0) + \frac{r^2}{a^2}\right] \end{aligned}$$

Neglecting the terms mentioned at the beginning, the expansion in a binomial series yields

[1] See A. Einstein, *Berl. Ber.*, 1914, 1058, equation (47a).

$$\frac{1}{R} = \frac{1}{a}\left\{1 + \frac{r}{a}(\sin\vartheta\cos\varphi\sin\vartheta_0 + \cos\vartheta\cos\vartheta_0) - \frac{1}{2}\frac{r^2}{a^2} + \frac{3}{2}\frac{r^2}{a^2}(\sin\vartheta\cos\varphi\sin\vartheta_0 + \cos\vartheta\cos\vartheta_0)^2\right\} \tag{9}$$

We denote the expression inside the curly bracket by K and write

$$\frac{1}{R} = \frac{K}{a} \tag{9a}$$

Substitution of (6), (7), (8), and (9a) into (3) yields

$$\gamma_{11}{}' = \frac{i\chi}{2\pi}\rho_0 a^3\omega^2\, da\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta\left(\frac{dx_4}{ds}\right)^3 \sin^3\vartheta\sin^2\varphi\, K$$

$$\gamma_{22}{}' = \frac{i\chi}{2\pi}\rho_0 a^3\omega^2\, da\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta\left(\frac{dx_4}{ds}\right)^3 \sin^3\vartheta\cos^2\varphi\, K$$

$$\gamma_{44}{}' = -\frac{i\chi}{2\pi}\rho_0 a\, da\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta\left(\frac{dx_4}{ds}\right)^3 \sin\vartheta\, K$$

$$\gamma_{12}{}' = -\frac{i\chi}{2\pi}\rho_0 a^3\omega^2\, da\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta\left(\frac{dx_4}{ds}\right)^3 \sin^3\vartheta\sin\varphi\cos\varphi\, K$$

$$\gamma_{14}{}' = \frac{\chi}{2\pi}\rho_0 a^2\omega\, da\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta\left(\frac{dx_4}{ds}\right)^3 \sin^2\vartheta\sin\varphi\, K$$

$$\gamma_{24}{}' = -\frac{\chi}{2\pi}\rho_0 a^2\omega\, da\int_0^{2\pi} d\varphi\int_0^{\pi} d\vartheta\left(\frac{dx_4}{ds}\right)^3 \sin^2\vartheta\cos\varphi\, K$$

$$\gamma_{13}{}' = \gamma_{23}{}' = \gamma_{33}{}' = \gamma_{43}{}' = 0 \tag{10}$$

The absolute value of the quantity dx_4/ds differs from unity only by terms of order $\omega^2 a^2$; it appears as a factor in the small first-order terms $\gamma_{\mu\nu}{}'$; hence it is sufficient to calculate them starting with the expression for the line element in the "zeroth" approximation:

$$ds^2 = -dx_1{}^2 - dx_2{}^2 - dx_3{}^2 - dx_4{}^2$$

$$\frac{ds^2}{dx_4{}^2} = -1 - \frac{dx_1{}^2 + dx_2{}^2 + dx_3{}^2}{dx_4{}^2}$$

$$= -1 + \omega^2 a^2 \sin^2\vartheta$$

$$\frac{ds}{dx_4} = i\left(1 - \frac{\omega^2 a^2}{2}\sin^2\vartheta\right)$$

$$\left(\frac{dx_4}{ds}\right)^3 = i\left(1 + \frac{3}{2}\omega^2 a^2 \sin^2\vartheta\right) \tag{11}$$

Because the accuracy of our calculation extends only to terms of order $\omega^2 a^2$, we can put $(dx_4/ds)^3 = i$ for all those $\gamma_{\mu\nu}'$ which already contain the factor ωa; merely for γ_{44}' will we use the expression (11). Furthermore, we put

$$\rho_0 da = \sigma$$

and then equations (10) become

$$\gamma_{11}' = -\frac{\chi}{2\pi}\sigma a^3\omega^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin^3\vartheta \sin^2\varphi\, K$$

$$\gamma_{22}' = -\frac{\chi}{2\pi}\sigma a^3\omega^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin^3\vartheta \cos^2\varphi\, K$$

$$\gamma_{44}' = \frac{\chi}{2\pi}\sigma a \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin\vartheta\, K \left(1 + \frac{3}{2}\omega^2 a^2 \sin^2\vartheta\right)$$

$$\gamma_{12}' = \frac{\chi}{2\pi}\sigma a^3\omega^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin^3\vartheta \sin\varphi\cos\varphi\, K$$

$$\gamma_{14}' = \frac{i\chi}{2\pi}\sigma a^2\omega \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin^2\vartheta \sin\varphi\, K$$

$$\gamma_{24}' = -\frac{i\chi}{2\pi}\sigma a^2\omega \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin^2\vartheta \cos\varphi\, K$$

$$\gamma_{13}' = \gamma_{23}' = \gamma_{33}' = \gamma_{34}' = 0 \tag{12}$$

If one substitutes the value of K from (9) into these expressions and calculates the integrals, one obtains

$$\gamma_{11}' = -\frac{\chi}{2\pi}\frac{M}{3a}a^2\omega^2\left(1 - \frac{r^2}{5a^2}\right)$$

$$\gamma_{22}' = -\frac{\chi}{2\pi}\frac{M}{3a}a^2\omega^2\left\{1 - \frac{r^2}{5a^2}\left(1 - 3\sin^2\vartheta_0\right)\right\}$$

$$\gamma_{44}' = \frac{\chi}{2\pi}\frac{M}{a}\left\{1 + a^2\omega^2\left[1 - \frac{r^2}{5a^2}\left(1 - \frac{3}{2}\sin^2\vartheta_0\right)\right]\right\}$$

$$\gamma_{24}' = -\frac{i\chi}{2\pi}\frac{M}{3a}\omega r\sin\vartheta_0$$

$$\gamma_{12}' = \gamma_{14}' = \gamma_{13}' = \gamma_{23}' = \gamma_{33}' = \gamma_{43}' = 0 \tag{13}$$

From these expressions one obtains the $\gamma_{\mu\nu}$ and subsequently the $g_{\mu\nu}$ by using the equations (1) and (2). Furthermore, we substitute in place of the polar coordinates r and ϑ_0 of the point under consideration its rectangular coordinates and replace the Einstein gravitational constant χ by the conventional one: $k = \chi/8\pi$ (velocity of light = 1). Then one gets

$$g_{11} = -1 - \frac{2kM}{a}\left\{1 + a^2\omega^2 - \frac{\omega^2}{10}(2z_0{}^2 + x_0{}^2)\right\}$$

$$g_{22} = -1 - \frac{2kM}{a}\left\{1 + a^2\omega^2 - \frac{\omega^2}{10}(2z_0{}^2 - 3x_0{}^2)\right\}$$

$$g_{33} = -1$$

$$g_{44} = -1 + \frac{2kM}{a}\left\{1 + \frac{5a^2\omega^2}{3} - \frac{\omega^2}{6}(2z_0{}^2 - x_0{}^2)\right\}$$

$$g_{24} = -i\frac{4kM}{3a}\omega x_0 \tag{14}$$

all remaining $g_{\mu\nu}$ vanish.

Now we want to free ourselves also from the special choice of the coordinate system. (We had placed the point under consideration in the Z-X plane.) For this purpose, we perform the transformation:

$$x_1' = x_1\cos\alpha + x_2\sin\alpha$$

$$x_2' = -x_1\sin\alpha + x_2\cos\alpha$$

$$x_3' = x_3$$

$$x_4' = x_4 \tag{15}$$

Then by means of the transformation law of a covariant tensor of second rank

$$g_{\sigma\tau}' = \frac{\partial x_\mu}{\partial x_\sigma'}\frac{\partial x_\nu}{\partial x_\tau'}g_{\mu\nu}$$

the coefficient scheme becomes

$$g_{\mu\nu} = \begin{Bmatrix} -1-\dfrac{2kM}{a}\left[1+a^2\omega^2-\dfrac{\omega^2}{10}(2z^2+x^2-3y^2)\right], & +\dfrac{2kM}{a}\dfrac{\omega^2}{5}xy, & 0, & +i\dfrac{4kM}{3a}\omega y \\ +\dfrac{2kM}{a}\dfrac{\omega^2}{5}xy, & -1-\dfrac{2kM}{a}\left[1+a^2\omega^2-\dfrac{\omega^2}{10}(2z^2-3x^2+y^2)\right], & 0, & -i\dfrac{4kM}{3a}\omega x \\ 0, & 0, & -1, & 0, \\ +i\dfrac{4kM}{3a}\omega y, & -i\dfrac{4kM}{3a}\omega x, & 0, & -1+\dfrac{2kM}{a}\left[1+\dfrac{5a^2\omega^2}{3}+\dfrac{\omega^2}{6}(2z^2-x^2-y^2)\right] \end{Bmatrix} \tag{16}$$

The index 0 in the coordinates is dropped here; in the following x, y, z denote the coordinates of the point under consideration.

B. Physical Part: The Motion of a Point Mass inside the Rotating Hollow Sphere. We want to set up the equations of motion of a point mass situated near the center of our rotating hollow sphere. The field in this neighborhood is characterized by the coefficient scheme of the $g_{\mu\nu}$ [equation (16) of the first section].

As is well known, the law of motion for a point mass in Einstein's theory is given by the condition $\delta \int ds = 0$, or, if one carries through the variation,[1]

$$\frac{d^2 x_\tau}{ds^2} = \Gamma^\tau_{\mu\nu} \frac{dx_\mu}{ds}\frac{dx_\nu}{ds}, \qquad \tau = 1 \ldots 4 \tag{17}$$

We have, according to the first type of approximation, for the "field components" $\Gamma^\tau_{\mu\nu}$

$$\Gamma^\tau_{\mu\nu} = -\begin{Bmatrix}\mu\nu \\ \tau\end{Bmatrix} = \begin{bmatrix}\mu\nu \\ \tau\end{bmatrix} = \frac{1}{2}\left(\frac{\partial g_{\tau\nu}}{\partial x_\mu} + \frac{\partial g_{\mu\tau}}{\partial x_\nu} - \frac{\partial g_{\mu\nu}}{\partial x_\tau}\right) \tag{18}$$

We merely want to consider motions of the point mass which are small with respect to the velocity of light, so that we can neglect the squares and products of the components of the velocity. Then one can cancel all those terms on the right-hand side of the equations (17) in which the index 4 does not appear, and, besides, we can substitute the derivatives with respect to s by the derivatives with respect to t. Taking into account that $dx_4/dt = i$, the equations (17) become

$$\frac{d^2 x_\tau}{dt^2} = 2i\left(\Gamma^\tau_{14}\frac{dx_1}{dt} + \Gamma^\tau_{24}\frac{dx_2}{dt} + \Gamma^\tau_{34}\frac{dx_3}{dt}\right) - \Gamma^\tau_{44} \tag{19}$$

Hence in what follows, only those of the field components $\Gamma^\tau_{\mu\nu}$ are taken into

[1] Indices appearing twice are to be summed over from 1 to 4.

consideration which contain the index 4 at least once. These are 16 quantities, which (though they are not tensorial components !) can be arranged in our case according to the scheme of an antisymmetric tensor of second rank. Since the partial derivatives with respect to x_4 vanish altogether in the stationary field, the quantities $\Gamma^{\tau}_{\sigma 4}$ can be written in the following way:[2]

$$
\begin{array}{ll}
\Gamma^1_{14} = 0, & \Gamma^1_{24} = \frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_2} - \frac{\partial g_{24}}{\partial x_1}\right), \\[2ex]
\Gamma^1_{34} = \frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_3} - \frac{\partial g_{34}}{\partial x_1}\right), & \Gamma^1_{44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_1} \\[2ex]
\Gamma^2_{14} = \frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_1} - \frac{\partial g_{14}}{\partial x_2}\right), & \Gamma^2_{24} = 0, \\[2ex]
\Gamma^2_{34} = \frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_3} - \frac{\partial g_{34}}{\partial x_2}\right), & \Gamma^2_{44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_2} \\[2ex]
\Gamma^3_{14} = \frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_1} - \frac{\partial g_{14}}{\partial x_3}\right), & \Gamma^3_{24} = \frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_2} - \frac{\partial g_{24}}{\partial x_3}\right), \\[2ex]
\Gamma^3_{34} = 0, & \Gamma^3_{44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_3} \\[2ex]
\Gamma^4_{14} = \frac{1}{2}\frac{\partial g_{44}}{\partial x_1}, & \Gamma^4_{24} = \frac{1}{2}\frac{\partial g_{44}}{\partial x_2}, \\[2ex]
\Gamma^4_{34} = \frac{1}{2}\frac{\partial g_{44}}{\partial x_3}, & \Gamma^4_{44} = 0
\end{array}
\tag{20}
$$

If one substitutes here the special values of the $g_{\mu\nu}$ from (16), one gets the following scheme:

[1]Note that this scheme corresponds to the six-vector $\mathfrak{M}$ of the electromagnetic field. The analogy between electrodynamics and (approximated) gravitational theory goes still further when one bears in mind that by the approximate integration the quantities $g_{14}, g_{24}, g_{34}, g_{44}$ are calculated in the same way in terms of the density and velocity of matter as the potentials $\mathfrak{A}_x$, $\mathfrak{A}_y$, $\mathfrak{A}_x$, Φ in terms of the electric four-current, and furthermore, that in our case the right-hand sides of equations (19) correspond altogether, except for numerical factors, to the components of the ponderomotive force $\mathfrak{G} + [\mathfrak{v}\mathfrak{H}]$!

$$\begin{matrix} 0 & i\dfrac{4kM}{3a}\omega & 0 & -\dfrac{kM}{3a}\omega^2 x \\ -i\dfrac{4kM}{3a}\omega & 0 & 0 & -\dfrac{kM}{3a}\omega^2 y \\ 0 & 0 & 0 & \dfrac{2kM}{3a}\omega^2 z \\ \dfrac{kM}{3a}\omega^2 x & \dfrac{kM}{3a}\omega^2 y & -\dfrac{2kM}{3a}\omega^2 z & 0 \end{matrix} \qquad (21)$$

Then, we obtain the equations of motion for our specific problem from (19) to (21):

$$\begin{aligned} \ddot{x} &= -\frac{8kM}{3a}\omega\dot{y} + \frac{kM}{3a}\omega^2 x \\ \ddot{y} &= \frac{8kM}{3a}\omega\dot{x} + \frac{kM}{3a}\omega^2 y \\ \ddot{z} &= -\frac{2kM}{3a}\omega^2 z \end{aligned} \qquad (22)$$

The right-hand sides of the equations represent the components of the force which our field exerts on the point mass with mass 1. As can be seen, the first terms of the X and Y components correspond to the Coriolis force and the second terms to the centrifugal force. At first sight, the third equation yields the surprising result that this "centrifugal force" possesses an axial component as well. Its occurrence in the field of the rotating sphere can be elucidated in the following way: From the viewpoint of the observer at rest, those surface elements of the hollow sphere situated near the equator have a larger velocity, and consequently a larger apparent (inertial and gravitating) mass, than those close to the poles. Hence the field of a rotating hollow sphere with a constant surface density corresponds to the field of a hollow sphere at rest with a surface density which increases with the polar angle ϑ. In the latter case, it is readily understandable that points lying outside the equatorial plane are dragged into it.

(Incidentally, one can also easily imagine that forces analogous to the centrifugal force occur in the interior of a hollow sphere endowed with such a nonuniform mass density. As is well known, in potential theory one can demonstrate the vanishing of the force field inside a hollow sphere with constant surface charge density as follows: The attractive force of the surface elements, which are seen from P under the angle $d\omega$ (see figure 1), is equal and opposite

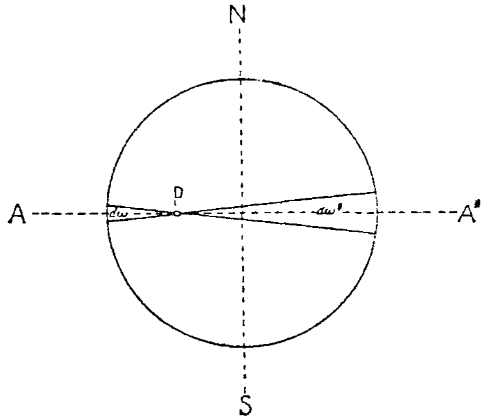

Fig. 1.

to the force exerted by those surface elements which lie within the opposite angle $d\omega'$. Of course, this no longer applies in the case of nonuniform surface density. Let AA' be the equatorial plane; then, with respect to the position of the point P as drawn in the figure, the surface elements lying inside the angle $d\omega$ are closer to the equator on the average, and are therefore specifically heavier than those of $d\omega'$. Hence there results a force in the direction of $A'A$, i.e., a tension that is perpendicular to the rotation axis, points away from it, and becomes smaller the closer the point P is shifted to the center.)

That we merely recognize a radial but never an axial component of the centrifugal force in nature could be reconciled with the results found here as follows: The approximation of the firmament by means of an infinitely thin hollow sphere is just incorrect. But even if we want to improve our approximation (say, by means of a spatial distribution of masses) we would never obtain a field equivalent to a real centrifugal field when using this method of integration. We would obtain such a field if we imagined all masses in the outer space (galaxies, etc.) as rotating and calculated their gravitational effect. But the solution for the retarded potentials [equation (3)] assumes the boundary condition $\lim \gamma_{\mu\nu} = 0$ for spatial infinity. As Einstein has shown in his cosmological paper,[1] these boundary conditions are approximately fulfilled for a coordinate system with respect to which the fixed stars are at rest on the average. Therefore, our solution (16) does not represent the field of a single rotating hollow sphere representing the Universe, rather the field inside of such a hollow sphere outside of which there are masses at still larger distances from the origin that are at rest on the average with respect to the chosen system of reference. For ex-

[1] *Berl. Ber.*, 1917, p. 142.

ample, the field as represented by equation (16) is that which would hold at the center of the Sun if instead of the Sun and all the planets a large hollow sphere existed, say, with the radius of Neptune's orbit and which rotated with an angular velocity ω with respect to the fixed stars. If there were observers at the center of this sphere, situated on a heavenly body, the gravitational field of which can be neglected and which rotates about the same axis with the angular velocity ω', the observers would perceive centrifugal and Coriolis forces consisting of the effect of their own rotation and the effect of the rotation of the hollow sphere. In the following, the influence of the field of the hollow sphere on the centrifugal force caused by its own rotation will be studied.

For this purpose, we introduce a coordinate system tied to the reference body rotating with the angular velocity ω'. This is performed by means of the transformation

$$\left.\begin{aligned} x' &= x \cos \omega' \frac{x_4}{i} + y \sin \omega' \frac{x_4}{i}, \qquad & z' &= z \\ y' &= -x \sin \omega' \frac{x_4}{i} + y \cos \omega' \frac{x_4}{i}, \qquad & x_4' &= x_4 \end{aligned}\right\} \tag{23}$$

By this transformation, the quantities of interest $g_{\mu 4}$ become

$$\begin{aligned} g_{14}' &= -iy'\left[\omega'\left(1 + \frac{2kM}{3a}\right) - \omega \frac{4kM}{3a}\right] \\ g_{24}' &= ix'\left[\omega'\left(1 + \frac{2kM}{3a}\right) - \omega \frac{4kM}{3a}\right] \\ g_{44}' &= -1 + \frac{2kM}{a}\left[1 + \frac{5a^2\omega^2}{3} - \frac{\omega^2}{3}z^2\right] + (x'^2 + y'^2) \\ &\quad \cdot \left\{\omega'^2\left(1 + \frac{2kM}{a}\right) - \omega\omega' \frac{4kM}{3a} + \omega^2 \frac{kM}{3a}\right\} \end{aligned} \tag{24}$$

If one constructs the equations of motion out of these quantities according to (19) and (20), one will get

$$\begin{aligned} \ddot{x} &= 2\left[\omega'\left(1 + \frac{2kM}{a}\right) - \omega \frac{4kM}{3a}\right]\dot{y} + \left\{\omega'^2\left(1 + \frac{2kM}{a}\right) - \omega\omega' \frac{4kM}{3a} + \omega^2 \frac{kM}{3a}\right\} x \\ \ddot{y} &= -2\left[\omega'\left(1 + \frac{2kM}{a}\right) - \omega \frac{4kM}{3a}\right]\dot{y} + \left\{\omega'^2\left(1 + \frac{2kM}{a}\right) - \omega\omega' \frac{4kM}{3a} + \omega^2 \frac{kM}{3a}\right\} y \\ \ddot{z} &= -\frac{2kM}{3a}\omega^2 z \end{aligned} \tag{25}$$

After setting $M = 0$ here, one obtains the usual centrifugal-Coriolis field:

$$\begin{aligned} \ddot{x} &= 2\omega'\dot{y} + \omega'^2 x \\ \ddot{y} &= -2\omega'\dot{x} + \omega'^2 y \\ \ddot{z} &= 0 \end{aligned} \tag{26}$$

After setting $M \neq 0$, $\omega = 0$, it becomes

$$\begin{aligned} \ddot{x} &= 2\omega'\left(1 + \frac{2kM}{a}\right)\dot{y} + \omega'^2\left(1 + \frac{2kM}{a}\right)x \\ \ddot{y} &= -2\omega'\left(1 + \frac{2kM}{a}\right)\dot{x} + \omega'^2\left(1 + \frac{2kM}{a}\right)y \\ \ddot{z} &= 0 \end{aligned} \tag{27}$$

wherefrom one can see how the inertial effects are influenced by the presence of the surrounding masses M. The centrifugal and Coriolis forces are multiplied by the factor $(1 + 2kM/a)$.

Finally, one can see from equation (25) that if the hollow sphere rotates in the same sense, this results in diminishing the centrifugal and Coriolis forces. When one sets

$$\omega' = \omega \frac{4kM}{3(2kM + a)} \tag{28}$$

then the Coriolis force vanishes. One could call the quantity $4kM/3(2kM + a)$ the dragging coefficient of the hollow sphere with respect to the Coriolis force. The centrifugal force cannot be made to vanish, since the expressions inside the curly brackets of equation (25), when set equal to zero, do not yield real roots for ω. In the reference system "at rest" ($\omega' = 0$), the expression for the centrifugal force was

$$\frac{kM}{3a}\omega^2(x^2 + y^2)^{1/2}$$

If one now allows the reference system to rotate in the same sense as the hollow sphere, the centrifugal force will at first decrease for small values of ω' and will reach a minimum when ω'/ω is equal to half the value of the "dragging coefficient.[1] Thereupon, it increases again and reaches the original value again which it had for $\omega' = 0$ as soon as ω'/ω is equal to the dragging coefficient. Then it will continue to increase along with ω', and for large ω' will reach a value hardly

[1] One can immediately convince oneself by differentiating the term inside the parenthesis.

different from the value it had in the absence of the hollow sphere [namely, $\omega'^2(x^2 + y^2)^{1/2}$], since according to our assumptions $2kM/a$ is small compared to 1.

At first sight, it seems contradictory to the spirit of a theory of relativity that the right-hand sides of the equations of motion (25) do not depend on the difference $\omega - \omega'$ alone. One should not forget, however, that we do not deal with only two bodies (point mass and hollow sphere) in the problem treated here; rather, owing to the boundary conditions $\lim \gamma_{\mu\nu} = 0$, still further distant masses at rest relative to the original reference system are introduced as a third element determining the field.

Summary

Using a concrete example, it is shown that forces occur in the (Einsteinian) gravitational field of distant rotating masses which are analogous to the centrifugal and the Coriolis forces, respectively. The peculiarities associated with this special case are discussed.

Vienna, December 1917, Institute for Theoretical Physics of the University.
(Received 21 December 1917.)

Correction to My Paper: "On the Effect of Rotating Distant Masses in Einstein's Theory of Gravitation"[1]

HANS THIRRING

Herr Laue and Herr W. Pauli kindly brought to my attention the following errors in my above-mentioned paper: In the approximate integration of the Einsteinian field equations, the quantity dV in the formulas for the retarded potentials $\gamma_{\mu\nu}$ [equation (3) *l.c.*] simply denotes the usual spatial volume element of integration space (in polar coordinates: $r^2 dr \sin\vartheta\, d\vartheta\, d\varphi$) and not, as erroneously claimed by me, the naturally measured volume element

$$i \frac{dx_4}{ds} r^2 dr \sin\vartheta\, d\vartheta\, d\varphi$$

[1] This journal [*Phys. Z.*], 19, 33, 1918.

Therefore, one has to drop a factor $i(dx_4/ds)$ everywhere in the equations (10) and (12). Furthermore, in going from (12) to (13) we incorrectly set $\int \rho_0 dV = M$. Instead, the correct formula is:

$$\int \rho_0 dV_0 = i \int \rho_0 \frac{dx_4}{ds} dV = M$$

or taking (11) into account,

$$M = 4\pi\sigma a^2 \left(1 + \frac{\omega^2 a^2}{3}\right)$$

By eliminating these errors, one obtains for the coefficients $g_{\mu\nu}$ of the line element, in place of (10), the scheme presented below [equation (16)] *:

$$g_{\mu\nu} = \left\{\begin{array}{cccc} -1 - \frac{2kM}{a}\left[1 + \frac{a^2\omega^2}{3} - \frac{2\omega^2}{15}(z^2 + x^2 - 2y^2)\right], & + \frac{2kM}{a}\frac{\omega^2}{5}xy, & 0, & +i\frac{4kM}{3a}\omega y \\ + \frac{2kM}{a}\frac{\omega^2}{5}xy, & -1 - \frac{2kM}{a}\left[1 + \frac{a^2\omega^2}{3} - \frac{2\omega^2}{15}(z^2 + y^2 - 2x^2)\right], & 0, & -i\frac{4kM}{3a}\omega x \\ 0, & 0, & 0, & 0 \\ +i\frac{4kM}{3a}\omega y, & -i\frac{4kM}{3a}\omega x, & 0, & -1 + \frac{2kM}{a}\left[1 + a^2\omega^2 - \frac{2\omega^2}{15}(2z^2 - x^2 - y^2)\right] \end{array}\right\} \tag{16}$$

The equations of motion for the point mass then read as follows:

$$\begin{aligned} \ddot{x} &= - \frac{8kM}{3a}\omega\dot{y} + \frac{4kM}{15a}\omega^2 x \\ \ddot{y} &= + \frac{8kM}{3a}\omega\dot{x} + \frac{4kM}{15a}\omega^2 y \\ \ddot{z} &= - \frac{8kM}{15a}\omega^2 z \end{aligned} \tag{22}$$

Hence the Coriolis force remains unchanged as compared to my original formula (22); however, for the centrifugal term one needs an additional factor of $\frac{4}{5}$.

Furthermore, the equations of motion with respect to the reference system rotating with angular velocity ω' read [see equation (25)]:

$$\ddot{x} = 2\left[\omega'\left(1 + \frac{2kM}{a}\right) - \omega\frac{4kM}{3a}\right]\dot{y} + \left\{\omega'^2\left(1 + \frac{2kM}{a}\right) - \omega\omega'\frac{8kM}{3a} + \omega^2\frac{4kM}{15a}\right\}x$$

*Translator's note: A typographical error should be corrected in equation (16): The entry for g_{33} must be "–1" instead of "0."

$$\ddot{y} = -2\left[\omega'\left(1+\frac{2kM}{a}\right)-\omega\frac{4kM}{3a}\right]\dot{x}+\left\{\omega'^2\left(1+\frac{2kM}{a}\right)-\omega\omega'\frac{8kM}{3a}+\omega^2\frac{4kM}{15a}\right\}y$$

$$\ddot{z} = -\frac{8kM}{3a}\omega^2 z \tag{25}$$

Also here the correction of the error simply yields an additional factor of $\frac{4}{5}$ in the terms with ω^2. Because of an additional mistake in the transformation to the rotating coordinate system, there originally occurred a factor of $\frac{4}{3}$ instead of $\frac{8}{3}$ in the term with $\omega\omega'$. Thus, two sentences in the next-to-last paragraph of my paper have still to be corrected as follows: "If one now allows the reference system to rotate in the same sense as the hollow sphere, the centrifugal force will at first decrease for small values of ω' and will reach a minimum when ω'/ω is equal to the 'dragging coefficient'. Thereupon, it increases again and reaches the original value which it had for $\omega' = 0$ as soon as ω'/ω is equal to twice the dragging coefficient."

The principal result of my paper (occurrence of centrifugal and Coriolis forces in the gravitational field of distant rotating masses) remains completely unchanged.

Vienna, 15 October 1920.

(Received 25 October 1980.)

On the Influence of the Proper Rotation of Central Bodies on the Motions of Planets and Moons According to Einstein's Theory of Gravitation

J. LENSE and H. THIRRING

In a paper published recently,[1] one of us has calculated approximately the field inside a rotating hollow sphere according to Einstein's theory of gravitation. This example seemed to be of principal interest in answering the question whether the rotation of distant masses according to Einstein's theory really produces a gravitational field equivalent to a "centrifugal field." Now it appears interesting, from another point of view, to consider the integration of the field

[1] Hans Thirring; this journal [*Phys. Z.*], **19**, 33, 1918; hereafter referred to as paper I.

equations for a rotating solid sphere, which can be simply carried out using the same methods. For, so long as one stood on the basis of Newton's theory, one could exactly substitute the field outside a sphere charged with uniform volume density (at rest or rotating) by the field of a material point of the same mass. Likewise, according to Einstein's theory, the field of an incompressible fluid sphere *at rest* is equivalent to that of a point mass,[1] but this is no longer valid for rotating spheres. As will be shown in what follows, additional terms appear here which correspond to centrifugal and Coriolis forces. Now since the planets move in the field of the Sun, which itself rotates, and the moons in turn in the field of the rotating planets, it does not seem excluded from the outset that one can obtain a novel confirmation of Einstein's theory by observing the perturbations caused by these additional terms. In the following, we calculate numerically the perturbations in the planetary orbital elements, which turn out to be below observational limits. For the moons of Jupiter, however, one obtains relatively large secular perturbations which should still lie within the errors of measurement.

§1. *The Calculation of the $g_{\mu\nu}$ for the Field of the Rotating Solid Sphere*

Notation:

l, radius of the sphere;
M, its mass;
ω, its angular velocity;
x', y', z', rectangular coordinates of a point of the integration space;
x, y, z, coordinates of the point under consideration;
k, gravitational constant;
ρ_0, naturally measured space density of matter.

The calculation is performed in analogy with that in the paper cited at the beginning: Einstein's method of approximate integration[2] is used, but this time, when forming the energy tensor of matter, the velocity of the masses producing the field is regarded as so small compared to 1 (velocity of light) that one can neglect the squares and products of the components of the velocity. (This is in contrast to the example treated in the previous paper. The terms of the centrifugal force, which are proportional to* w^2, are omitted and only the Coriolis terms appear.) Neglecting such terms is completely justified when one considers that $l\omega$ for the Sun and all the planets is very small in a system of units in which the velocity of light = 1. Besides, in the case treated here we consider the field at a larger distance from the surface of the sphere. Let r denote the distance be-

[1] K. Schwarzschild, *Berl. Ber.*, 1916, p. 424.
[2] A. Einstein, *Berl. Ber.*, 1916, p. 688.
*Translator's note: A typographical error should be corrected here. Read "ω^2" instead of "w^2."

tween the point under consideration and the center of the sphere, r' the distance between the center and the integration element, and R the distance between the point under consideration and the integration element, then we expand $1/R$ in a series in r'/r up to the second order.

Then, exactly as in paper I, we start from the approximate solution given by Einstein[1]:

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}, \qquad \delta_{\mu\nu} \begin{aligned} &= 1, \quad \mu = \nu \\ &= 0, \quad \mu \neq \nu \end{aligned}$$

$$\gamma_{\mu\nu} = \gamma'_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu} \sum_{\alpha} \gamma'_{\alpha\alpha} \tag{1}$$

$$\gamma'_{\mu\nu} = -\frac{k}{2\pi} \int \frac{T_{\mu\nu}(x', y', z', t - R)}{R} dV_0$$

Thereafter, we form the energy tensor of stress-free matter

$$T_{\mu\nu} = T^{\mu\nu} = \rho_0 \frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = \sigma_0 \frac{dx_\mu}{dx_4}\frac{dx_\nu}{dx_4}\left(\frac{dx_4}{ds}\right)^2 \tag{2}$$

with the expressions for the components of velocity:

$$\begin{aligned} \frac{dx_1}{dx_4} &= -i\frac{dx'}{dt} = ir'\omega \sin\vartheta' \sin\varphi' \\ \frac{dx_2}{dx_4} &= -i\frac{dy'}{dt} = -ir'\omega \sin\vartheta' \cos\varphi' \\ \frac{dx_3}{dx_4} &= 0 \end{aligned} \tag{3}$$

(r', ϑ', φ' polar coordinates of a point of the sphere; the rotation is performed about the Z axis), and by neglecting the terms with ω^2 we obtain

$$T_{\mu\nu} = \rho_0 \left(\frac{dx_4}{ds}\right)^2 \cdot \left\{ \begin{matrix} 0, & 0, & 0, & ir'\omega \sin\vartheta' \sin\varphi' \\ 0, & 0, & 0, & -ir'\omega \sin\vartheta' \cos\varphi' \\ 0, & 0, & 0, & 0 \\ ir'\omega \sin\vartheta' \sin\varphi', & -ir'\omega \sin\vartheta' \cos\varphi', & 0, & 1 \end{matrix} \right. \tag{4}$$

[1] In the corresponding equation (2) of paper I, the factor $\delta_{\mu\nu}$ has been inadvertently omitted.

According to equations (7) and (8) of paper I, we have to set for dV_0

$$dV_0 = i\,\frac{dx_4}{ds}\,r'^2\,dr'\,\sin\vartheta'\,d\vartheta'\,d\varphi' \tag{5}$$

In order to express $1/R$ in terms of the integration variables, we choose the coordinate system again such that the point under consideration is placed in the Z-X plane. Then, by introducing polar coordinates,

$$x = r\sin\vartheta, \qquad y = 0, \qquad z = r\cos\vartheta$$

hold and one has

$$\begin{aligned} R^2 &= (r'\sin\vartheta'\cos\varphi' - r\sin\vartheta)^2 + r'^2\sin^2\vartheta'\sin^2\varphi' + (r'\cos\vartheta' - r\cos\vartheta)^2 \\ &= r^2\left[1 - \frac{2r'}{r}(\sin\vartheta'\cos\varphi'\sin\vartheta + \cos\vartheta'\cos\vartheta) + \frac{r'^2}{r^2}\right] \end{aligned}$$

Upon expanding in binomial series up to second term,

$$\begin{aligned} \frac{1}{R} = \frac{1}{r}\Bigg\{1 &+ \frac{r'}{r}(\sin\vartheta'\cos\varphi'\sin\vartheta + \cos\vartheta'\cos\vartheta) - \frac{1}{2}\frac{r'^2}{r^2} \\ &+ \frac{3}{2}\frac{r'^2}{r^2}(\sin\vartheta'\cos\varphi'\sin\vartheta + \cos\vartheta'\cos\vartheta)^2\Bigg\} \end{aligned} \tag{6}$$

We again denote the expression inside the curly bracket by K and write

$$\frac{1}{R} = \frac{K}{r} \tag{6a}$$

If we now substitute (4), (5), and (6a) into the last equation (1), then we obtain

$$\begin{aligned} \gamma'_{44} &= -\frac{i\chi}{2\pi}\frac{\rho_0}{r}\int_0^l r'^2\,dr'\int_0^{2\pi} d\varphi'\int_0^{\pi} d\vartheta'\left(\frac{dx_4}{ds}\right)^3 \sin\vartheta'\,K \\ \gamma'_{14} &= \frac{\chi}{2\pi}\frac{\rho_0}{r}\,\omega\int_0^l r'^3\,dr'\int_0^{2\pi} d\varphi'\int_0^{\pi} d\vartheta'\left(\frac{dx_4}{ds}\right)^3 \sin^2\vartheta'\sin\varphi'\,K \\ \gamma'_{24} &= -\frac{\chi}{2\pi}\frac{\rho_0}{r}\,\omega\int_0^l r'^3\,dr'\int_0^{2\pi} d\varphi'\int_0^{\pi} d\vartheta'\left(\frac{dx_4}{ds}\right)^3 \sin^2\vartheta'\cos\varphi'\,K \\ \gamma'_{11} &= \gamma'_{22} = \gamma'_{33} = \gamma'_{12} = \gamma'_{13} = \gamma'_{23} = \gamma'_{34} = 0 \end{aligned} \tag{7}$$

By neglecting the ω^2 terms and using the first type of approximation, we have

$$\left(\frac{dx_4}{ds}\right)^3 = i$$

[See equation (11) of paper I.] When one substitutes this value for $(dx_4/ds)^3$ as well as the expression for K from (6) and (6a) into (7), one obtains after evaluation of the integrals

$$\begin{aligned}
\gamma'_{44} &= \frac{\chi}{2\pi}\frac{M}{r} \\
\gamma'_{14} &= 0 \\
\gamma'_{24} &= -i\,\frac{\chi}{2\pi}\frac{M}{r}\frac{l}{5r}\,\omega l \sin\vartheta \\
\gamma'_{11} &= \gamma'_{22} = \gamma'_{33} = \gamma'_{12} = \gamma'_{13} = \gamma'_{23} = \gamma'_{34} = 0
\end{aligned} \tag{8}$$

Moreover, it follows from (1) that once rectangular coordinates are introduced and the Einsteinian gravitational constant is replaced by the Newtonian one $k = \chi/8\pi$,

$$\begin{aligned}
g_{11} &= g_{22} = g_{33} = -1 - \frac{2kM}{r} \\
g_{44} &= -1 + \frac{2kM}{r} \\
g_{24} &= -i\,\frac{4kM}{r}\frac{lx}{5r^2}\,\omega l \\
g_{12} &= g_{13} = g_{23} = g_{14} = g_{34} = 0
\end{aligned} \tag{9}$$

If now by a rotation of the system one frees oneself from the special choice of the coordinates in which the point under consideration is situated in the Z-X plane, one obtains the final scheme for the coefficients:

$$g_{\mu\nu} = \left\{\begin{matrix}
-1 - \dfrac{2kM}{r}, & 0, & 0, & i\,\dfrac{4kM}{5r}\dfrac{ly}{r^2}\,\omega l \\[2ex]
0, & -1 - \dfrac{2kM}{r}, & 0, & -i\,\dfrac{4kM}{5r}\dfrac{lx}{r^2}\,\omega l \\[2ex]
0, & 0, & -1 - \dfrac{2kM}{r}, & 0 \\[2ex]
i\,\dfrac{4kM}{5r}\dfrac{ly}{r^2}\,\omega l, & -i\,\dfrac{4kM}{5r}\dfrac{lx}{r^2}\,\omega l, & 0, & -1 + \dfrac{2kM}{r}
\end{matrix}\right. \tag{10}$$

§2. *The Equations of Motion of a Point Mass in the Field of a Rotating Solid Sphere*

In the following, the equations of motion of a point mass in the field of a rotating solid sphere will be set up and we will assume that its velocity is so small that we can neglect the squares and the products of the components of velocity with respect to 1. Thereby, we emphasize from the outset that we are only interested in finding those *perturbation terms* of the planetary motion which are due to the rotation of the central bodies. In order to obtain a sufficiently exact solution of the planetary problems in accordance with Einstein's theory, one still has to add to the perturbation terms calculated here those which lead to the known perihelion precession.[1] The terms due to the proper rotation of the central bodies already emerge from the first approximation of Einstein's theory, while the perihelion perturbation just mentioned was only obtained in the second approximation. Nevertheless, it is still not permissible to take the former into account and to neglect the latter. The reason why one is not allowed to do that results from the following considerations: The ansatz for the force developed below and the Newtonian one differ by additional terms which are proportional to $\omega l v$, where v represents the velocity of the planets or the moons, respectively, while ωl represents the velocity of a point on the equator of the central body. But now the inequality

$$v > \omega l \tag{11}$$

is valid for both the Sun–planets system as well as the planets–moons systems that will be considered. If we take the terms with $\omega l v$ into account, we have all the more reason to consider those terms in the equations of motion which contain the squares and products of the components of velocity of the point mass. But if we do that, then we are not at all allowed to calculate with the first approximation alone, for those terms which are added in the second approximation to the Newtonian terms are in ratio to the former as $\alpha/r : 1$ ($\alpha = 2kM$). But the square of the velocity of a planet is also of the same order as α/r; thus taking the quadratic terms in the velocities into account logically requires the consideration of terms emerging from the second approximation. It follows therefrom that the calculations performed here would not actually make sense owing to the validity of the inequality (11). Nevertheless, they can be used in practice if one keeps in mind that all perturbations considered here are so small as to allow us to superpose them linearly. One bases the computation on the equations of motion as given by Einstein in his Mercury paper and adds the perturbation terms calculated in what follows. In this way, one arrives at the desired result which is the calculation of an orbit including relativistic effects.

[1] A. Einstein, *Berl. Ber.*, 1915, p. 831.

As shown in paper I, by neglecting the above-mentioned terms and with the coordinates $x_1 = x, x_2 = y, x_3 = z, x_4 = it$ the general equations of motion

$$\frac{d^2 x_\tau}{ds^2} = \Gamma^\tau_{\mu\nu} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds}$$

become

$$\frac{d^2 x_\tau}{dt^2} = 2i\left(\Gamma^\tau_{14} \frac{dx_1}{dt} + \Gamma^\tau_{24} \frac{dx_2}{dt} + \Gamma^\tau_{34} \frac{dx_3}{dt}\right) - \Gamma^\tau_{44} \tag{12}$$

The 16 quantities $\Gamma^\tau_{\sigma 4}$ that appear here are, according to the first type of approximation for a stationary field, given by

$$\begin{aligned}
&\Gamma^1_{14} = 0, && \Gamma^1_{24} = \frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_2} - \frac{\partial g_{24}}{\partial x_1}\right),\\
&\Gamma^1_{34} = \frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_3} - \frac{\partial g_{24}}{\partial x_1}\right), && \Gamma^1_{44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_1}\\
&\Gamma^2_{14} = \frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_1} - \frac{\partial g_{14}}{\partial x_2}\right), && \Gamma^2_{24} = 0,\\
&\Gamma^2_{34} = \frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_3} - \frac{\partial g_{34}}{\partial x_2}\right), && \Gamma^2_{44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_2}\\
&\Gamma^3_{14} = \frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_1} - \frac{\partial g_{14}}{\partial x_3}\right), && \Gamma^3_{24} = \frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_2} - \frac{\partial g_{24}}{\partial x_3}\right),\\
&\Gamma^3_{34} = 0, && \Gamma^3_{44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_3}\\
&\Gamma^4_{14} = \frac{1}{2}\frac{\partial g_{44}}{\partial x_1}, && \Gamma^4_{24} = \frac{1}{2}\frac{\partial x_{44}}{\partial x_2},\\
&\Gamma^4_{34} = \frac{1}{2}\frac{\partial g_{44}}{\partial x_3}, && \Gamma^4_{44} = 0
\end{aligned} \tag{13}$$

For our field, as given by the equations (10), this scheme becomes*

*Translator's note: The entry for Γ^3_{14} in (14) must have "r^2" in the denominator instead of "z^2."

$$
\begin{matrix}
0, & -i\,\dfrac{2kM}{5r^2}\,\dfrac{\omega l^2}{r}\,\dfrac{x^2+y^2-2z^2}{r^2}, & -i\,\dfrac{6kM}{5r^2}\,\dfrac{\omega l^2}{r}\,\dfrac{yz}{r^2}, & \dfrac{kM}{r^2}\,\dfrac{x}{r} \\[2ex]
+i\,\dfrac{2kM}{5r^2}\,\dfrac{\omega l^2}{r}\,\dfrac{x^2+y^2-2z^2}{r^2}, & 0, & +i\,\dfrac{6kM}{5r^2}\,\dfrac{\omega l^2}{r}\,\dfrac{xz}{r^2}, & \dfrac{kM}{r^2}\,\dfrac{y}{r} \\[2ex]
+i\,\dfrac{6kM}{5r^2}\,\dfrac{\omega l^2}{r}\,\dfrac{yz}{z^2}, & -i\,\dfrac{6kM}{5r^2}\,\dfrac{\omega l^2}{r}\,\dfrac{xz}{r^2}, & 0, & \dfrac{kM}{r^2}\,\dfrac{z}{r} \\[2ex]
-\dfrac{kM}{r^2}\,\dfrac{x}{r}, & -\dfrac{kM}{r^2}\,\dfrac{y}{r}, & -\dfrac{kM}{r^2}\,\dfrac{z}{r}, & 0
\end{matrix}
\tag{14}
$$

By substituting these values for the $\Gamma^{\tau}_{\sigma 4}$ into (12), we obtain the desired equations of motion*:

$$
\begin{aligned}
\ddot{x} &= \frac{kM}{r^2}\,\frac{\omega l^2}{r}\left[\frac{4}{5}\,\frac{x^2+y^2-2z^2}{r^2}\,\dot{}\; + \frac{12}{5}\,\frac{yz}{r^2}\,\dot{z}\right] - \frac{kM}{r^2}\,\frac{x}{r} \\
\ddot{y} &= -\frac{kM}{r^2}\,\frac{\omega l^2}{r}\left[\frac{4}{5}\,\frac{x^2+y^2-2z^2}{r^2}\,\dot{x} + \frac{12}{5}\,\frac{z}{r^2}\,\dot{z}\right] - \frac{kM}{r^2}\,\frac{y}{r} \\
\ddot{z} &= \frac{kM}{r^2}\,\frac{\omega l^2}{r}\,\frac{12}{5}\,\frac{z}{r}\,\frac{x\dot{y}-y\ddot{x}}{r} - \frac{kM}{r^2}\,\frac{z}{r}
\end{aligned}
\tag{15}
$$

The last terms on the right-hand side represent the Newtonian force; as discussed above, one has to substitute in its place the force components according to Einstein's Mercury paper. The first terms on the right-hand side are the perturbation terms of interest caused by the proper rotation of the central bodies.

§3. *Calculation of the Perturbations due to the Proper Rotation of the Central Body*

The perturbation terms appearing in the equations (15) have to be regarded as components X, Y, Z of the perturbing force caused by the proper rotation of the central bodies. We decompose them into three other mutually perpendicular components S, T, W, where S denotes the radial one, T the transversal one, and W the orthogonal one (i.e., normal to the plane of the planetary orbit) and we introduce the following conventional astronomical notation:

*Translator's note: In the first of equations (15), a "y" is missing in "$\dot{y}$", in the second, an "x" is missing in "xz/r^2", and in the last one, "$z//r$" and "$\ddot{x}$" must be replaced by "z/r" and "$\dot{x}$," respectively.

a, semimajor axis;
e, eccentricity;
$p = a(1 - e^2)$, semilatus rectum;
$i = \sphericalangle\, y\Omega\Pi$, inclination;
$\Omega = \sphericalangle\, XO\Omega$, longitude of the node;
$\tilde{\omega}$ = broken $\sphericalangle\, XO\Pi$, longitude of the pericenter;
L_0, mean longitude of the epoch, i.e., mean longitude of the planet or satellite at time $t = 0$ (also a broken angle, measured from the X axis);
$v = \sphericalangle\, \Pi OP$, true anomaly;
$u = \sphericalangle\, \Omega OP = v + \tilde{\omega} - \Omega$, argument of the latitude;
U, period of revolution of the planet or satellite in days;
$n = 2\pi/U = (kM/a^3)^{1/2}$, mean daily motion;
$C = r^2 \dot{v} = na^2(1 - e^2)^{1/2}$, twice the areal velocity.

Furthermore, we abbreviate the constant appearing in the equations (15) by setting $4kM\omega l^2/5 = K$.

Now we have*

$$x = r(\cos u \cos \Omega - \sin u \sin \Omega \cos i)$$

$$y = r(\cos u \sin \Omega + \sin u \cos \Omega \cos i)$$

$$z = r \sin u \sin i$$

$$r = \frac{p}{1 + e \cos v}$$

$$x\dot{y} - y\ddot{x} = C \cos i$$

Moreover,

$$S = X(\cos u \cos \Omega - \sin u \sin \Omega \cos i) + Y(\cos u \sin \Omega + \sin u \cos \Omega \cos i) + Z \sin u \sin i$$

$$T = -X(\sin u \cos \Omega + \cos u \sin \Omega \cos i) - Y(\sin u \sin \Omega - \cos u \cos \Omega \cos i) + Z \cos u \sin i$$

$$W = X \sin \Omega \sin i - Y \cos \Omega \sin i + Z \cos i$$

If one substitutes the values of X, Y, Z obtained from equations (15) into these formulas for S, T, W by using the given relations and notations, one finds after a longer intermediary calculation

*Translator's note: In the fifth formula below, read "$\dot{x}$" instead of "$\ddot{x}$."

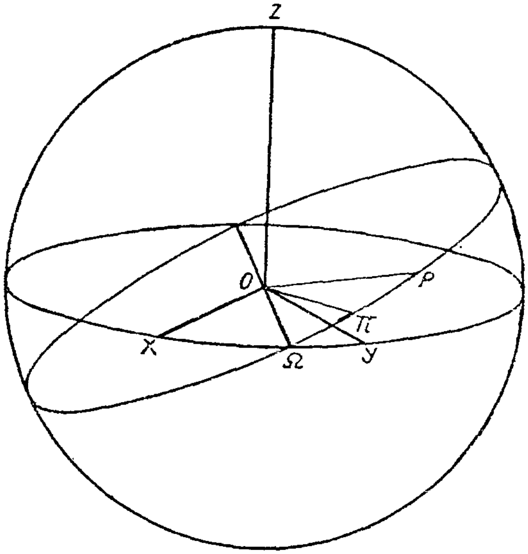

Fig. 2. Π and P denote the positions of the pericenter and of the planet or satellite, respectively, which are projected from the middle point O of the central body onto the sphere.

$$
\begin{aligned}
S &= \frac{KC \cos i}{r^4} \\
T &= -\frac{K\dot{r} \cos i}{r^3} = -\frac{KCe \cos i \sin v}{pr^3} \\
W &= \frac{K \sin i}{r^4}(2C \sin u + r\dot{r} \cos u) \\
&= \frac{KC \sin i}{r^4}\left(\frac{re \sin v \cos u}{p} + 2 \sin u\right)
\end{aligned}
\tag{16}
$$

The variation of the orbital elements caused by the perturbing force is given by the equations

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{n(1-e^2)^{1/2}}\left(Se \sin v + T\frac{p}{r}\right) \\
\frac{de}{dt} &= \frac{(1-e^2)^{1/2}}{na}\left[S \sin v + T\left(e + \frac{r+a}{a}\cos v\right)\right] \\
\frac{di}{dt} &= \frac{1}{C} Wr \cos u
\end{aligned}
$$

$$\frac{d\Omega}{dt} = \frac{1}{C \sin i} Wr \sin u$$

$$\frac{d\tilde{\omega}}{dt} = \frac{(1-e^2)^{1/2}}{nae}\left[- S \cos v + T\left(1 + \frac{r}{p}\right) \sin v\right] + 2 \sin^2 \frac{i}{2} \frac{d\Omega}{dt}$$

$$\frac{dL_0}{dt} = - \frac{2}{na^2} Sr + \frac{e^2}{1+(1-e^2)^{1/2}} \frac{d\tilde{\omega}}{dt} + 2(1-e^2)^{1/2} \sin^2 \frac{i}{2} \frac{d\Omega}{dt}$$

which, by substituting the values (16), can be represented in the following form:

$$\frac{da}{dt} = 0$$

$$\frac{de}{dt} = \frac{K \cos i}{Ca} \sin v \cdot \dot{v}$$

$$\frac{di}{dt} = \frac{K \sin i}{Cp} \cos u \; [e \sin v \cos u + 2(1 + e \cos v) \sin u] \; \dot{v}$$

$$\frac{d\Omega}{dt} = \frac{K}{Cp} \sin u [e \sin v \cos u + 2(1 + e \cos v) \sin u] \; \dot{v}$$

$$\frac{d\tilde{\omega}}{dt} = - \frac{K \cos i}{Ca}\left(2 + \frac{1+e^2}{e} \cos v\right) \dot{v} + 2 \sin^2 \frac{i}{2} \frac{d\Omega}{dt}$$

$$\frac{dL_0}{dt} = - \frac{2K \cos i}{na^2 p}(1 + e \cos v)\, \dot{v} + \frac{e^2}{1+(1-e^2)^{1/2}} \frac{d\tilde{\omega}}{dt} + 2(1-e^2)^{1/2} \sin^2 \frac{i}{2} \frac{d\Omega}{dt}$$

In the sense of perturbation theory, we regard the orbital elements on the right-hand side as constants due to the extraordinarily small factors K. We only integrate over v, i.e., we calculate the perturbations to first order in K, and observe that $u = v + \tilde{\omega} - \Omega$. Moreover, if we introduce $K_1 = K/na^3$, we obtain

$$\Delta a = 0$$

$$\Delta e = - \frac{K_1 \cos i}{(1-e^2)^{1/2}} \cos v$$

$$\Delta i = - \frac{K_1 \sin i}{2(1-e^2)^{3/2}} (\cos 2u + 2e \cos v \cos^2 u)$$

$$\Delta\Omega = \frac{K_1}{(1-e^2)^{3/2}}\left[v - \frac{1}{2} \sin 2u + e\left(\sin v - \frac{1}{2} \sin 2u \cos v\right)\right]$$

$$\Delta\tilde{\omega} = - \frac{K_1 \cos i}{(1-e^2)^{3/2}}\left(2v + \frac{1+e^2}{e} \sin v\right) + 2 \sin^2 \frac{i}{2} \Delta\Omega$$

$$\Delta L_0 = -\frac{2K_1 \cos i}{1-e^2}(v + e \sin v) + \frac{e^2}{1+(1-e^2)^{1/2}} \Delta\tilde{\omega} + 2(1-e^2)^{1/2} \sin^2 \frac{i}{2} \Delta\Omega$$

There follows the interesting result that the perturbations in the semimajor axis vanish exactly. While only periodic terms appear in Δe and Δi, additional secular terms occur in the perturbation of the other elements; namely, because of $v = nt$ + periodic terms,

$$\begin{aligned} \Delta\Omega &= \frac{K_1}{(1-e^2)^{3/2}} nt \\ \Delta\tilde{\omega} = \Delta L_0 &= -\frac{2K_1}{(1-e^2)^{3/2}}\left(1 - 3\sin^2\frac{i}{2}\right) nt \end{aligned} \tag{17}$$

§4. *Numerical Results*

The numerical evaluation shows that these secular perturbations on the Sun-planets system even for the period of one century are beyond any observational possibility; for they reach a maximum of 0.01″ (for the perihelion of Mercury). Things are different in the planet-moons systems. Here somewhat larger numbers are encountered. It is better for the numerical calculation to transform the formulas (17). Let us introduce the following notation:

l, radius of the planet in centimeters;
τ, period of rotation of the planet in days;
a, semimajor axis of the satellite in centimeters;
a_1, semimajor axis of the planetary orbit in centimeters;
U, orbital period of the satellite in days;
U_1, orbital period of the planet in days;
J, number of days in a year;
c, velocity of light in cm sec^{-1}.

Then the formulas resulting from (17)

$$2\Delta\Omega = -\Delta\tilde{\omega} = -\Delta L_0 = \frac{\pi^2 J}{9c^2}\frac{l^2}{\tau U^2} \tag{18}$$

yield, in units of arc seconds per century, the perturbations in the elements of the satellite orbit due to the rotation of the planet. We have set $e^2 = i^2 = 0$, which is allowed for the moons involved within the desired accuracy.

In addition, the perturbations are superposed additively in the sense of Section 2 with those discussed by Einstein in his Mercury paper, which are partly due to the direct effect of the planet and partly due to the perturbing effect of the Sun. The former are given by

$$\Delta\Omega = 0, \qquad \Delta\tilde{\omega} = \Delta L_0 = \frac{5\pi^2 J}{24c^2} \frac{a^2}{U^3(1 - e^2)} \tag{19}$$

and the latter[1] by*

$$4\Delta\Omega = \Delta\tilde{\omega} = \Delta L_0 = \frac{5\pi^2 J}{12c^2} \frac{a_1{}^2}{U_1{}^3} \tag{20}$$

both in arc seconds per century. In equation (20), the eccentricity and the inclination of the planetary and satellite orbits were neglected again due to the fact that these terms are extremely small as shown in Table I. They are considerably smaller for all remaining moons.

The perturbations due to the proper rotation of the planet are contained in Table II. They remain below 0.5″ for all other satellites.

The largest terms are analogous to the Einsteinian perihelion motion of Mercury [formulas (19)], as emerges from Table III.† They are smaller than 0.5″ for the moons that are left out.

If we now add the three types of terms in order to obtain all relativistic effects, the following has to be taken into account: The correction of Newton's

Table I

	$\Delta\Omega$	$\Delta\tilde{\omega} = \Delta L_0$
Moon	+1.9″	+7.7″
Both moons of Mars	+0.7″	+2.7″

Table II

	Jupiter			Saturn				
	V	I	II	1	2	3	4	5
$\Delta\Omega$	+1′53″	+ 9″	+2″	+20″	+10″	+ 5″	+2″	+1″
$\Delta\tilde{\omega} = \Delta L_0$	−3′46″	−18″	−4″	−41″	−19″	−10″	−5″	−2″

[1] W. de Sitter, Planetary motion and the motion of the moon according to Einstein's theory, *Amsterdam Proc.*, **19**, 1916. In the formulas (20), XY plane denotes the plane of planetary orbit. In de Sitter's paper, on p. 379 formula (38), the factor $\frac{1}{4}$ is missing in front of $\delta\tilde{\omega}$.

*Translator's note: The result of de Sitter cited in footnote 1 is, in fact, correct. Hence equation (20) gives values of $\Delta\tilde{\omega}$ and ΔL_0 too large by a factor of 4. Tables I and IV must be corrected accordingly.

†Translator's note: The existence of a satellite of Saturn, mentioned as "Saturn 10" or "Themis" in Tables III and IV, was announced by Pickering in 1905. This has not been confirmed by subsequent observations.

Table III[a]

		$\Delta\tilde{\omega} = \Delta L_0$		$\Delta\tilde{\omega} = \Delta L_0$
Mars	1	22″	Jupiter I	4′28
	2	2	II	1 24
Saturn	1	5′46″	III	26
	2	3 03	IV	6
	3	1 47	V	36 37
	4	59	Uranus 1	22
	5	25	2	10
	6	3	3	3
	7	2	4	1
	10	2	Neptune's moon	5

[a] $\Delta\Omega = 0$.

law treated in Einstein's Mercury paper is caused by a perturbing force acting along the radius vector. Its components, according to the cited paper, amount to

$$S = -\frac{3n^2a^3C}{2c^2}\frac{\dot{v}}{r^2}, \qquad T = W = 0$$

so they are independent of the choice of the coordinate system. Consequently, the corresponding perturbations [formulas (19) and Table III] can be related to an arbitrary XY plane. However, the variations of the elements contained in the formulas (20) and caused by the perturbing effect of the Sun (representing the deviation from the classical form) are referred to the orbital plane of the planet as already mentioned. The same holds for the numbers in Table I, whereas in Table II, which contains the perturbation terms caused by the rotation of the planets, everything refers to the equatorial plane of the central body in accordance with the choice of the coordinate system in the present paper. Therefore, the following is valid for Table IV, which summarizes all the relativistic effects: Terms (19) and (20) only appear in connection with the Moon and the two moons of Mars; hence, the plane of reference is the orbital plane of the planet. In the case of the satellites of Jupiter and Saturn, however, the reference plane is the equatorial plane of the corresponding central body since here again only terms (18) and (19) appear. The perturbations of the moons of Uranus and the moon of Neptune contain only terms (19); consequently, the plane of reference can be chosen arbitrarily.

The following can be stated about the column "Δt": The secular perturbation in the mean longitude causes a variation in the mean daily motion. That is, in the absence of relativistic effects a certain correction should be made for the time that elapses between two definite events, such as the darkenings of the moons of Jupiter. This correction for a time span of a hundred years is given in

Table IV

	$\Delta\Omega$	$\Delta\tilde{\omega} = \Delta L_0$	Δt
Moon	2″	8″	13.9s
Mars 1. Phobos	1	25	0.5
2. Deimos	1	5	0.4
Jupiter I	9	4′ 10″	29.5
II	2	1 20	18.9
III	0	26	12.5
IV	0	6	7.1
V	1′ 53″	32 51	1m 5.4s
Saturn 1. Mimas	20	5 05	19.2
2. Enceladus	10	2 44	15.0
3. Tethys	5	1 37	12.2
4. Dione	2	54	9.2
5. Rhea	1	23	6.9
6. Titan	0	3	3.3
7. Hyperion	0	2	2.7
10. Themis	0	2	2.9
Uranus 1. Ariel	0	22	3.7
2. Umbriel	0	10	2.7
3. Titania	0	3	1.5
4. Oberon	0	1	1.0
Neptune's moon	0	5	2.1

the last column of Table IV and can be obtained by means of the formula

$$\Delta t = \frac{1}{15} U \Delta L_0$$

Summary

The perturbation terms in the orbits of the planets and moons which, according to Einstein's theory, are caused by the proper rotation of the central body turn out to be smaller than those which emerge from the second approximation of the theory and lead to the perihelion motion of Mercury. They need not at all be taken into consideration in comparison with the latter terms in the case of the planetary orbits; however, for the orbits of the moons of Jupiter and Saturn they have to be taken into account. The secular perturbations due to all relativistic effects involved were calculated for the moons of the exterior planets. Though in some cases—for example, for the 5th moon of Jupiter—they reach a considerable magnitude, the present observations should not be precise enough as to allow a test of the theory using the perturbations of the orbits of the moons.

Vienna, February 1918. Institute for Theoretical Physics of the University.
(Received 21 February 1918.)

§(3): *Discussion of Thirring's Papers*

Hans Thirring investigated the gravitational "magnetic" field generated by a rotating hollow sphere. However, Mach's ideas on the origin of inertia provided the main impetus for his work. Mach criticized Newton's formulation of the laws of mechanics since the motion of matter was referred to an absolute space. This notion is related to an ensemble of global inertial frames each in uniform motion with respect to the others. A body in uniform motion persists in that state and offers "resistance" when accelerated with respect to absolute space. This characteristic property of matter is embodied in the notion of inertial mass. An observer can determine if matter is accelerated with respect to inertial frames since extra inertial forces (proportional to the inertial mass) appear in an accelerated frame. The success of Newtonian mechanics has to be reconciled with the fact that only the motion of matter with respect to other "matter" is physically detectable.

Newton considered the introduction of the abstract notion of absolute space necessary for the mathematical formulation of the laws of mechanics. On the contrary, Mach considered all motion to be relative. In rejecting the notion of absolute space Mach had predecessors in Leibniz and Berkeley, among others. If only relative motion has significance, the inertial frames must be determined by matter.

To give these vague ideas a more definite formulation, one may extend the principle of relativity to accelerated motion and postulate that inertial forces are due to the gravitational *field* generated by all the matter in the universe. According to Einstein's relativistic theory of gravitation (which has observational support for macroscopic phenomena), however, these notions must be rejected since they imply the *global equivalence* of inertial and certain gravitational forces in contrast to Einstein's principle of equivalence which is purely local. To illustrate this point, consider a variant of Newton's bucket experiment in which the bucket is uniformly accelerated. Other than forces of electromagnetic origin (such as viscosity), the fluid in the bucket is also subject to a *uniform* inertial force field (relative to the bucket). A contradiction arises, however, if the bucket is now treated as freely falling in the gravitational field generated by all the matter in the universe in accelerated motion, since according to Einstein's theory the only external gravitational forces that affect the motion of the fluid relative to the bucket are *tidal* forces.[1] This example also illustrates why the case of rotational acceleration is interesting: The (first-order) tidal and centrifugal forces have similar functional forms. It is clear that these arguments are based on the tensor character of the gravitational field in Einstein's theory and are completely independent of any boundary conditions on the space-time metric at spatial infinity.

[1] This argument against the relativity of accelerated motion can be formulated equally well within the framework of Newton's theory of gravitation.

Moreover, the gravitational interaction of the distant matter in the universe cannot be responsible for the inertial mass if Einstein's principle of equivalence is valid [2].[2]

Let us return to Mach's criticism of Newton's absolute space: Matter moves with respect to absolute space but cannot affect it. Mach's solution was that the configuration of distant matter in the universe determines the characteristics of this inertial space. The local metrical properties of space-time are related to the distribution of matter according to Einstein's theory of gravitation. Furthermore, this theory transforms the question of the inertial properties of our local space-time into a problem of modern cosmology. An important requirement for our local frame to be inertial is the local isotropy of space. This has been tested by several null experiments of great precision [4, 5]. The precise relationship between our local inertial frame and the rest frame determined by distant matter is a difficult problem of observational astronomy. It follows from the study of the motions of the inner planets of the solar system that the two frames do not rotate with respect to each other with an uncertainty of ~1 second of arc per century [6-8]. The discoveries of Hubble expansion and the cosmic microwave background radiation have opened up new possibilities for the discussion of this problem. Severe restrictions may be placed on the rotation of the universe from the recent studies of the isotropy of the microwave background radiation. Moreover, the dipole temperature anisotropy of this radiation may be used to determine the linear motion of our local frame with respect to the rest frame of the radiation [9].

The major contribution of the work of Thirring (and of Lense and Thirring) was the demonstration of how the local inertial frames are affected by the motion of nearby matter ("dragging of the inertial frames"). It is interesting to elucidate this issue in a coordinate invariant manner in relation to the work of Thirring. Consider first a freely falling "laboratory" frame in an external field and assume that a Fermi frame [10] is set up along the geodesic path of a representative point in the laboratory ("center of mass"). Let τ be the proper time along this path and let $\mathbf{\Omega}_D(\tau)$ be the rate of rotation of the locally nonrotating frame with respect to a certain local fiducial frame (e.g., the rest frame of the laboratory). If it is established that this latter frame does not rotate with respect to the rest frame of inertial observers at spatial infinity, $\mathbf{\Omega}_D$ corresponds to the dragging of the local inertial frames with respect to those of infinity. The motion of free particles with respect to the Fermi frame may be described to first order by

[2] In the theories of Newton and Einstein, the strength of the gravitational interaction is constant in space and time. On the basis of Mach's ideas regarding the origin of inertia, a decrease in the inertial mass may, however, be expected as a result of the Hubble expansion of the universe. This may be interpreted as a decrease in the "constant" of gravitation on a cosmological time scale. There is no firm evidence at present for such a variation [3].

$$\frac{d^2 X^i}{d\tau^2} + K^i{}_j(\tau)\, X^j = 0$$

where X^i are the (Fermi) coordinates of a particle in the neighborhood of the representative point. Here $K_{ij}(\tau)$ is a symmetric traceless matrix denoting certain ("electric") components of the Riemann tensor evaluated in the Fermi frame. To express this equation of motion with respect to the fiducial frame which rotates with frequency $\mathbf{\Omega} = -\mathbf{\Omega}_D(\tau)$ with respect to the Fermi frame, let $x_i = M_{ij}(\tau)X^j$ be the relevant coordinate transformation and

$$\frac{dM_{ij}}{d\tau} = e_{ikl}\Omega_l M_{kj}$$

where e_{ijk} is the alternating symbol with $e_{123} = 1$. The equation of motion of a free particle with respect to the fiducial frame is then

$$\frac{d^2 x_i}{d\tau^2} + 2e_{ijk}\,\Omega_j \frac{dx_k}{d\tau} + \left(\Omega_i\Omega_j - \delta_{ij}\Omega^2 + e_{ilj}\frac{d\Omega_l}{d\tau} + k_{ij}\right) x_j = 0$$

where $k_{ij} = M_{im}M_{jn}K_{mn}$.

Thirring considers a hollow sphere of mass M and radius a rotating uniformly with angular frequency ω. The shell has uniform mass density so that its angular momentum is $J = \frac{2}{3} Ma^2 \omega$. The effect of rotation on the gravitational field in the interior of the sphere is considered to first order in the Newtonian constant k and to second order in ω. A test particle at rest at the center of the sphere follows a geodesic C in this field. The Fermi frame along C rotates uniformly in the same sense as the hollow sphere with a frequency

$$\Omega_D = \frac{2kJ}{a^3} = \frac{4kM}{3a}\,\omega$$

with respect to the rest frame of the inertial observers at infinity. In a local frame that rotates with frequency $\mathbf{\Omega} = -\mathbf{\Omega}_D$ with respect to the Fermi frame, the equation of motion of a free test particle near the center of the hollow sphere can be written as above with k_{ij} determined simply by

$$k_{ij}(\tau) = -\frac{1}{2}\left(\frac{\partial^2 g_{44}}{\partial x_i \partial x_j}\right)_C$$

Thirring's calculation of the metric tensor to second order in ω implies that k_{ij} is diagonal with $k_{11} = k_{22} = -(4kM/15a)\,\omega^2$ and $k_{33} = -2k_{11}$. The equation of motion comprising the dragging of the inertial frames and the gravitational tidal force of the rotating mass is the same as that given by Thirring, as expected.[3] Thirring neglected any stresses in the shell which, however, is not in general permissible due to the conservation laws of energy and momentum [11, 12]. When due account is taken of elastic stresses in the rotating shell, only terms in

[3]Hence Thirring's interpretation of the ω^2-terms as analogous to centrifugal forces cannot be maintained, even apart from the fact that gravitational forces can only be locally identified with inertial forces.

the metric tensor proportional to ω^2 turn out to be different from those given by Thirring; consequently, the proper tidal matrix is in fact just $\frac{1}{2}$ of that obtained from Thirring's results.

The weak-field result of Thirring for the dragging of the inertial frames is valid in the limit $kM \ll a$; its generalization to strong fields has been discussed by a number of authors [13–17]. These studies provide further insight into the dependence of Ω_D/ω on the nature of the matter distribution.

To summarize, Mach's ideas provided strong stimuli for the development of the relativistic theory of gravitation by subordinating the existence of inertia to the matter distribution and by suggesting the possible equivalence of inertia and gravitation. Following Einstein [18], attempts have been made to use extensions of Mach's ideas to solve the problem of boundary conditions in relativistic cosmology. The gravitational field equations are local differential equations for the metrical properties of space-time. Once the matter content has been specified, boundary conditions are in general necessary to provide a unique solution for the cosmic gravitational field. To avoid this problem, it is tempting to postulate, for instance, that the universe is spatially closed. It appears, however, that a much more extensive knowledge of the matter and radiation content throughout the universe is necessary before the problem of the cosmic field can be adequately tackled.[4]

§(4): *Discussion of the Paper of Lense and Thirring*

Josef Lense and Hans Thirring investigated the influence of the gravitational "magnetic" field of a rotating body on the motion of test masses in orbit around it. The gravitational field of a rotating body is characterized by a "vector potential," which in the first approximation corresponds to the gravitational "magnetic" dipole due to the mass current. The vector potential is given by[5,6]

$$\mathbf{G} \approx -2k \frac{\mathbf{J} \times \hat{\mathbf{r}}}{r^2}$$

for $r \gg 2kM$ and $r \gg A$, where $A = J/M$ is the angular momentum per unit mass of the body. The precession of a test gyroscope (i.e., a test gravitational "magnetic" dipole) at rest outside a matter distribution indicates the dragging of the inertial frames or, equivalently, the presence of the gravitational "magnetic"

[4] Review papers on Mach's ideas [19–21] may be consulted for more extensive discussions and further references to the literature.

[5] For a localized matter distribution, the "vector" potential is defined by $G_i = -g_{i4}/g_{44}$ in a system of Schwarzschild-like coordinates.

[6] Lense and Thirring calculated the gravitational "vector" potential for a *rotating solid sphere* in the weak-field approximation. In this procedure, deviations of the source from spherical symmetry are naturally reflected in the vector potential in a form similar to the multipole expansions of electrodynamics [22, 23].

field $-2\boldsymbol{\Omega}_D$,

$$\boldsymbol{\Omega}_D = -\frac{1}{2}\nabla \times \mathbf{G} = \frac{kJ}{r^3}[3(\hat{\mathbf{r}}\cdot\hat{\mathbf{J}})\hat{\mathbf{r}} - \hat{\mathbf{J}}]$$

The weak-field analogy with electrodynamics provides insight into the gravitational field of a rotating body.[7] This analogy is incomplete, however, in contrast to the notion of the dragging of the inertial frames which is of general validity. To illustrate this point, consider a test gyroscope with its center of mass at rest in the stationary field of a Kerr black hole [26]. Let **S** be the spin vector of the gyroscope with respect to a local orthonormal tetrad frame whose spatial axes correspond to the spherical coordinate axes of static observers at infinity. The gyroscope precesses according to the formula

$$\frac{d\mathbf{S}}{d\tau} = \boldsymbol{\Omega}_D \times \mathbf{S}$$

where the precession frequency is given by

$$\boldsymbol{\Omega}_D = f(r,\vartheta)\,\nabla\times\left(\frac{k\mathbf{J}\times\hat{\mathbf{r}}}{r^2}\right) + g(r,\vartheta)\,\frac{kM}{r}\cdot\frac{(k\mathbf{J}\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{r^3}$$

in Boyer-Lindquist coordinates [27]. The functions f and g are determined from

$$\Sigma^{3/2}(\Sigma - 2kMr)f = r^4\Delta^{1/2}$$

$$kMg = (\Delta^{-1/2}\Lambda - r)f$$

where $\Sigma = r^2 + A^2\cos^2\vartheta$, $\Lambda = r^2 - A^2\cos^2\vartheta$, and $\Delta = r^2 + A^2 - 2kMr$. The radial coordinate ranges from the static limit

$$r = kM + (k^2M^2 - A^2\cos^2\vartheta)^{1/2}$$

to infinity. The weak-field results are obtained for $r \gg 2kM$,

$$f(r,\vartheta) = 1 + \frac{kM}{r} + \cdots$$

$$g(r,\vartheta) = 1 + \frac{5}{2}\left[1 - \frac{1}{5}\,\frac{A^2}{k^2M^2}(1 + 2\cos^2\vartheta)\right]\frac{kM}{r} + \cdots$$

while f and g both diverge as the static limit surface is approached. At this surface, the assumption that the gyroscope's center of mass is at rest as determined by the inertial observers at infinity breaks down. Despite the dramatic increase

[7]It is interesting to note that the formal analogy with electrodynamics, pointed out by Einstein [24], was explored by Thirring [25], who confirmed that in the weak-field limit the geodesic equation may be written in the form of the Lorentz force law in terms of certain gravitational "electric" and "magnetic" fields.

in the magnitude of precession frequency close to the static limit surface, the general direction of this vector is similar to its weak-field limit: At the poles, $\mathbf{\Omega}_D$ is parallel to $\mathbf{J}$ and is given by $2kr^{-3}f(r, 0)\,\mathbf{J}$, but as ϑ varies, its direction gradually deviates away from $\mathbf{J}$ until it reverses at the equator, where $\mathbf{\Omega}_D = -kr^{-3}(1 - 2kM/r)^{-1}\,\mathbf{J}$. Thus, even in the strong-field region, the gravitational influence of a rotating mass is characterized by the dragging of the inertial frames.

In the weak-field approximation a test body follows a perturbed Keplerian orbit about a central rotating mass. To characterize the motion, one may use instead of the position and velocity of the test mass at any given moment of time the six orbital parameters of the osculating ellipse which the test particle would follow if the perturbing force were turned off at that instant. In terms of these parameters, the equations of motion then reduce to the Lagrange "planetary" equations. Lense and Thirring found that the perturbing effect of the gravitational "magnetic" field produced no change in the semimajor axis (a) of the osculating ellipse, while the eccentricity (e) and the inclination (i) of the orbit were affected only in a periodic manner, so that they remained unchanged when averaged over time. The only secular effects appeared in the longitude of the ascending node, the longitude of the pericenter, and the mean longitude. They discovered that the average rate of advance of the ascending node was given by

$$\mathbf{\Omega}_{\mathrm{LT}} = \frac{2k\mathbf{J}}{a^3(1 - e^2)^{3/2}}$$

while the longitude of the pericenter, as well as the mean longitude, varied on the average at a rate $(1 - 3\cos i)\,\Omega_{\mathrm{LT}}$. It follows from these results that the average rate of change of the longitude of the pericenter relative to the longitude of the node is given by

$$-3\Omega_{\mathrm{LT}}(\hat{\mathbf{J}} \cdot \hat{\mathbf{L}})\,\hat{\mathbf{L}}$$

where $\mathbf{L}$ is the orbital angular momentum and $\hat{\mathbf{J}} \cdot \hat{\mathbf{L}} = \cos i$. Thus the frequency of pericenter precession is on the average given by

$$\boldsymbol{\omega}_{\mathrm{LT}} = \Omega_{\mathrm{LT}}[\hat{\mathbf{J}} - 3(\hat{\mathbf{J}} \cdot \hat{\mathbf{L}})\,\hat{\mathbf{L}}]$$

which also characterizes the precession of the orbit as a whole. This result may be expressed as follows [28]: For an unperturbed orbit, the orbital angular momentum and the Lenz vectors are constants of the motion that characterize the orientation of the orbit. The rotation of the central mass produces on the average a precessional motion in these vectors with a frequency $\boldsymbol{\omega}_{\mathrm{LT}}$, which therefore corresponds to the average precession frequency of the elliptical orbit.[8]

The pericenter precession due to the rotation of the central mass was first

[8]Note that the average frequency of precession of the orbital angular momentum is given, equally well, by $\mathbf{\Omega}_{\mathrm{LT}}$.

considered for an equatorial orbit by de Sitter [29]. Lense and Thirring provided a general treatment valid for any orbital inclination.[9] This precession effect has been utilized in the theoretical determination of the angular momentum of the Kerr black hole [31, 32].

The observation of the Lense–Thirring precession in the orbits of the planets and moons could provide a new significant test of Einstein's theory in the solar system. This possibility was investigated in considerable detail by Lense and Thirring, who pointed out that the effects were too small to be measurable. This conclusion is equally valid today, although new possibilities have been opened up by the advent of space exploration.

To measure the Lense–Thirring precession, an orbit must have considerable eccentricity or inclination; furthermore, the observations must be so accurate as to allow the extraction of this small effect from the competing—and generally more significant—Newtonian and relativistic precessions. The multipole moments of the central masses and many-body effects account for the main Newtonian component, while the relativistic part is due to the purely spherical post-Newtonian fields of the Sun and the planets. The main effects of this post-Newtonian field are Einstein's pericenter precession and de Sitter's three-body effect due to the tidal influence of the Sun on a planet-moon orbit. Lense and Thirring discussed the relativistic perturbations in detail and provided estimates of the known significant effects, namely, Einstein's pericenter precession, the Lense–Thirring orbital precession, and de Sitter's three-body effect.[10] Moreover, they drew attention to the fact that for a given period of time (e.g., a century), the most significant relativistic orbital precession effect in the solar system is the Einstein precession for the moons of Jupiter and Saturn. The prospects for the observation of this effect have been reexamined recently [34]. A long-range observing program with improved accuracy is, in any case, necessary. The recent discovery of a new satellite of Jupiter and several new inner moons of Saturn by the Voyager spacecraft [35, 36] is encouraging, since the relativistic effects are expected to be even greater for these satellites. It is a cherished hope that, as space exploration continues, the various relativistic orbital effects can become measurable in the not-too-distant future.

The artificial satellites provide a new tool in attempts to test the predictions of Einstein's theory. Ginzburg [37] considered the possibility of measuring the perigee shift of an artificial satellite due to the rotation of the Earth, thereby providing a test of the Lense–Thirring pericenter precession. On the other hand, an interesting experiment has been suggested by Van Patten and Everitt [38, 39]

[9] In spite of a claim to the contrary [30], the Lense–Thirring results correctly give the frequency of the pericenter motion for any inclination angle i, including $i = \pi$.

[10] A summary of the various relativistic effects in the orbits of the planetary satellites has been given by Lense [33]. In terms of the general order of magnitude, the Einstein precession is the largest for these orbits, followed by the Lense–Thirring and de Sitter effects, respectively.

for the measurement of the Lense-Thirring precession of the orbital angular momentum. In this experiment, the dragging of the line of nodes is measured by placing two counter-orbiting drag-free spacecraft in polar orbits about the Earth.[11]

An alternative approach to the problem of determining the gravitational "magnetic" field of a rotating mass was considered by Schiff [41] and independently by Pugh. It consists in measuring the precession rate of a gyroscope in Earth orbit due to the coupling of the gyroscope's angular momentum to the rotation of the Earth. An experiment of this type using a superconducting gyroscope has been under active development at Stanford University by Fairbank, Everitt and their collaborators [42].

The Stanford gyroscope experiment and the Van Patten-Everitt counter-orbiting satellites experiment are the only two significant "laboratory" experiments thus far proposed to measure the dragging of the inertial frames due to a rotating mass [3].

Note Added in Proof

In a recent paper, V. B. Braginsky and A. G. Polnarev [*JETP Lett.*, **31**, 415 (1980)] have suggested a new experiment to measure the gravitational influence of a rotating mass. In this experiment, the contribution of the gravitational "magnetic" field of the Earth to the relative tidal acceleration experienced by masses at the ends of a spring in orbit around the Earth is measured. This idea has been discussed in detail by B. Mashhoon and D. S. Theiss [*Phys. Rev. Lett.*, **49**, 1542 (1982)], and the various general relativistic effects have been calculated. To distinguish the effect of the gravitational magnetic field from inertial effects, a local inertial frame along the orbit must be precisely defined. In fact, local gyroscopes must satisfy essentially the same performance criteria as in the Stanford gyroscope experiment.

Furthermore, a new gravitational effect of a rotating mass has recently been found by B. Mashhoon and D. S. Theiss. See B. Mashhoon, *Gen. Rel. Grav.*, March, 1984.

Acknowledgments

It is a pleasure to thank Professor J. Lense for helpful comments. The work of one of the authors (B.M.) was supported by the Alexander von Humboldt Foundation.

[11] In addition, a preliminary analysis [40] indicates that the Van Patten-Everitt experiment may also be used, by means of satellite-to-satellite range-rate measurements, to search for gravitational waves that are incident on the Earth and have a frequency in the range 10^{-4} Hz $\lesssim \nu \ll 10$ Hz.

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